THE MATHEMATICAL GAZETTE

EDITED BY
T. A. A. BROADBENT, M.A.

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The Mathematical Association.

THE ANNUAL MEETING will be held at the London Day Training College, Southampton Row, London, W.C. 1, on *Thursday, 5th January*, 1933, at 2.30 p.m., and *Friday, 6th January*, 1933, at 10 a.m. and 2.30 p.m.

Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 15s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette."

Change of Address should be notified to a Secretary. If Copies of the "Gazette" fail for lack of such notification to reach a member, duplicate copies can be supplied only at the published price.

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THE CONGRESS AT ZÜRICH.

A NOT unnatural consequence of my having complied with a request to act as a Delegate of the Mathematical Association at the International Congress of Mathematicians, which was held at Zürich during the week September 4-12, is that I have been asked to write a short account of the Congress for the *Gazette*. It is a little difficult to avoid giving merely an epitome of the official programme; and my difficulties are somewhat increased by my having, like many others, deliberately abstained from attending the Congresses at Strassburg and Toronto in 1920 and 1924, for reasons which are now, happily, a matter of past history, and by my having been unavoidably prevented from going to Bologna in 1928; in fact my only previous experience of a Congress was gained at Cambridge in 1912.

A rough count of the official list gave the number of members of the Congress as about 680 (though a few of these were not present), and they were accompanied by about 180 members of their families; so far as I could ascertain, not more than about 150 out of the 680 were Swiss. The corresponding figures for the Congress at Bologna were 836 members, accompanied by 280 members of their families; and 336 out of the 836 were Italians. In view of the economic circumstances, it is a little remarkable that the number of foreign mathematicians at Zürich should exceed the number at Bologna, though one can see that there was a tendency for people to economise at the expense of their families.

This is the first occasion on which the Congress has paid a return visit, the first Congress having been held at Zürich in 1897; there were then 204 members accompanied by 38 ladies, and about seventeen of the members of the first Congress, including the President of the first Congress, Prof. Geiser, also attended the Congress in 1932.

A substantial change in procedure has been made since 1928; it has been decided not to print all communications in full, but to issue two comparatively small volumes of Proceedings only; the first will contain the records of the meetings and the formal lectures, while the second will contain abstracts (limited to about 300 words) of other

communications. While this change may be partly due to financial considerations, it certainly seems desirable for the reason that, for ease of reference, mathematicians ought to be encouraged to publish their most important work in standard periodicals which are to be found in all important libraries rather than in Proceedings of Congresses which are apt to become exceedingly scarce. It may be a relief to some mathematicians to realise that they are not going to receive anything resembling the six substantial volumes which followed the Congress at Bologna.

The Congress opened with an informal reception at the Studentenheim on the Sunday evening, and more formally with addresses of welcome by the President of the Organizing Committee, Prof. Fueter, and by representatives of the Canton on Monday morning. The mornings throughout the week were devoted to formal lectures in the Technische Hochschule, the lecturers being Fueter (Theory of Ideals), Carathéodory (Functions of several complex variables), Julia (Development of the theory of functions of a complex variable), Pauli (Mathematical methods of Quantum Mechanics), Tschebotaröw (Galois Theory), Carleman (Linear integral equations), Cartan (Riemannian spaces), Bieberbach (Fundamental regions in the theory of functions), Morse (Calculus of variations), Noether (Hypercomplex systems), Bohr (Almost periodic functions), Severi (Functions of several variables and Algebraic Geometry), R. Nevanlinna (Riemann Surfaces), Wavre (Planetary figures), Alexander (Topology), F. Riesz (Derivates of functions of a real variable), Valiron (The Borel-Julia theorem on meromorphic functions), Sierpinski (Sets of points which are effectively definable), Bernstein (Aleatory quantities), Menger (Modern methods and problems in Geometry), Stenzel (Modes of thought in Greek Mathematics) ; a lecture by Hardy on Additive Theory of Numbers was announced, but was not given.

Four afternoons were occupied at the University by the meetings of the Sections, of which there were ten (1, Algebra and Theory of Numbers ; 2, Analysis ; 3, Geometry ; 4, Probability ; 5, Mechanics and Physics ; 6, Astronomy ; 7, Engineering ; 8, Logic and Philosophy ; 9, History ; 10, Pedagogy) ; Sections 2, 3 and 6 were further subdivided. At these meetings, communications were limited to 15 minutes, and the number of papers read in all the Sections was nearly 200. Among the English mathematicians who read papers were Hardy, Linfoot, Milne-Thomson, Mordell, Neville, Paley, Winn, and Miss Cartwright. Section 2 was the most popular with about 60 papers, while Sections 3 and 5 had over thirty each, Section 1 about twenty, Sections 4 and 8 had about a dozen each, and Section 6 about eight. Sections 7 and 10 were apparently represented by one paper each, and I did not discover Section 9. I got the impression that the Committee transferred papers from Section 2 to Section 1 whenever practicable in order to avoid a still greater preponderance of papers in Section 2 ; at any rate, rather to my surprise, I found on most afternoons that I would feel rather more at home in Section 1 than in any of the subdivisions of Section 2.

The Congress inevitably suffered from two disadvantages; limitations of time usually made it necessary for two lectures to be given simultaneously in the mornings (though the actual selection of simultaneous lectures had evidently been carefully arranged by the Committee so as to cause as little inconvenience as possible), and, of course, there were usually at least half a dozen Sectional meetings taking place simultaneously in the afternoons. The other disadvantage was that due to the confusion of tongues; a fair number of the speakers bore in mind the fact that a large proportion of their audience was listening to a foreign language and took care to speak as clearly as possible (for example, I was able to follow the lectures of Carathéodory and Julia almost as well as if they had been speaking in English), but others, I am sure, were comprehensible only to their own countrymen.

The atmosphere of the Sectional meetings (as opposed to the formal Lectures) tended to be a little frigid; discussion was invited on each paper, but often nobody said anything, and then the Chairman usually called on the author of the next paper; at one meeting which I attended, however, Hahn, who was in the Chair, made a short commendatory speech after each paper, and the audience's appreciation of his action seemed to me to be an ample justification of the trouble which he must have taken in discovering something appropriate to say on each paper.

Enough has now been said concerning what took place inside the lecture rooms, and I now turn to the pleasanter side of the week. On Tuesday afternoon the official delegates were all invited to tea by Herr von Schulthess-Bodmer and his wife at their estate on the island (or rather peninsula) of Au, about a dozen miles from Zürich across the lake; some of the party stayed on the peninsula rather longer than they had originally intended. Thursday was devoted to excursions, of which four were arranged, a drive to the Klausen Pass, the ascents of Pilatus and the Rigi, and a steamboat trip across Lake Lucerne from Flüelen (at the foot of the ascent to the St. Gotthard tunnel) to Lucerne. On Monday there was a concert at the Tonhalle (I did not attend it, since, like the late Master of Christ's, I am "immune to music"). On Saturday there was a social gathering at the Municipal Theatre; we were there entertained by speeches by members of the Federal Council; these were followed by a ballet, an informal dinner and a dance; in connection with the dinner we were each presented with a book of coupons (presumably for the committee to be able to check the amounts consumed with the amounts supplied with the caterers), and the effect of seeing in this book possibilities of all the things which we might have was really overwhelming. A very eminent mathematician (I hold him in too much respect to mention his name) told me that this was one of the few books of which he had read every word from cover to cover, his wonder increasing with every page that he turned. Coupons 1-4 each provided a course; 5, 8, 11, 12, 15 and 17-19 were merely advertisements of the various caterers (but they all added to the effect); 6 and 7 each produced a half-bottle of wine; 9 and 10 respectively

told one that coffee and beer could be obtained *ad lib.* (a slight snag here was that the programme stated that those who wanted beer were requested to go to the Cellar-Restaurant for it); 13, 14 and 16 provided respectively a liqueur, fruit or cheese, and pastry. Finally it was requested that we should not dance except on the stage. While on the subject of food, I might mention that many of us made our first acquaintance with the American cafeteria system at lunch at the Studentenheim, and some of us, when confronted with the necessity of getting a whole meal on to a tray, made hopelessly bad shots as to the amounts that we would require, some in excess, others in defect.

For most of us the Congress ended on the Sunday afternoon when we were entertained to tea at the Hotel Dolder by representatives of the city of Zürich; there was, however, a party which visited the Observatory on the Jungfraujoch on the Monday and Tuesday.

It is understood that the next Congress, in 1936, will be held at Oslo, the first visit to a Scandinavian country; there are probably many of us who have already decided to go there to renew the friendships which were made at Zürich and which will have to be kept up by correspondence in the meantime.

I cannot conclude without expressing my appreciation of the excellence of the organisation of the Congress and the hospitality of our hosts; in particular, English members of the Congress have special cause to be grateful to Dr. Gut who, whenever advice or information was wanted, seemed certain to be found in the Reception Room, ready to resolve one's difficulties in a moment.

G. N. W.

GLEANINGS FAR AND NEAR.

877. A story is told by Einstein against himself. . . . He is not a ready reckoner, and during the crisis of the mark he thought that a tram conductor had given him back too much change—a hundred thousand marks or so—and had to be convinced of his error. " Everybody ", said the kindly tram conductor, " hasn't the gift of calculation with these big figures. I mustn't take advantage of you."—H. G. Wells, *The Work, Wealth and Happiness of Mankind*, p. 680 (1932). [Per Prof. E. H. Neville.]

878. If it be true that there is some number m such that the equation of the $(m-1)$th degree is algebraically resoluble while the equation of the mth degree is not algebraically resoluble, it must be because the second problem is not *ejusdem generis* with the first; and consequently that there must be some simple and elementary way of stating the first problem which is not algebraically applicable to describe the second.—C. J. Hargreave, Introduction to *Essay on the Resolution of Algebraic Equations* (1866). [Per Prof. E. H. Neville.]

879. On peut regarder la mécanique comme une géométrie à quatre dimensions, et l'analyse mécanique comme une extension de l'analyse géométrique.—J. L. Lagrange, *Théorie des Fonctions Analytiques*, partie iii, ch. i (Second edition, 1813; 1). [Per Prof. E. H. Neville.]

SIMILAR FIGURES.

By A. van der Meersch, A.C.G.I.

If it was required to divide the area in Fig. 1 into n equal parts—say, for simplicity, three equal parts—how should we proceed ?

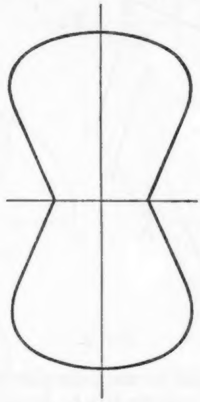


Fig. 1.

There is a method, given in books on geometry, of constructing a figure similar to a given figure and n times its area. This method pivots on a property of similar figures as follows : if *ABCDE* and

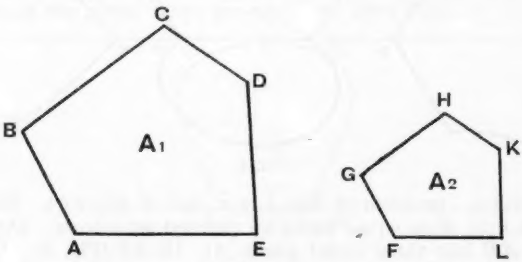


Fig. 2.

FGHKL are similar figures on bases *AE* and *FL* respectively, then

$$A_1 : A_2 = AE^2 : FL^2.$$

For purposes of comparison with what follows, the solution by this method is here recalled.

To describe a figure similar to the given figure *ABCDEF* and n times its area, we describe a similar figure on base *Ab* where $Ab^2 = n \,.\, AB^2$. To determine the position of b we draw a semi-

circle of diameter $JG=n$, mark off $JH=1$, and let a perpendicular HK to JG meet the circle in K. Then we have

$$JK^2=JH \cdot JG=n.$$

Drawing APQ such that $AP=1$ and $AQ=JK$, the position of b is constructed at once.

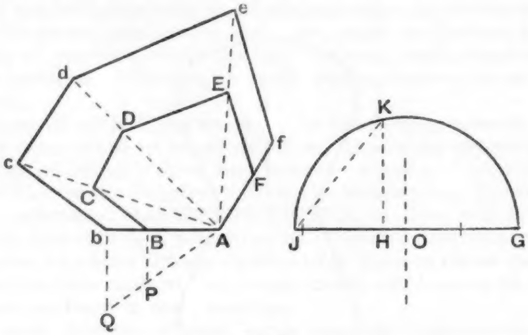


FIG. 3.

Against this, it is possible to use a simpler method, *in the numerous cases where a median line can readily be found*, that is, a line dividing the figure into two equal parts as in Fig. 4.

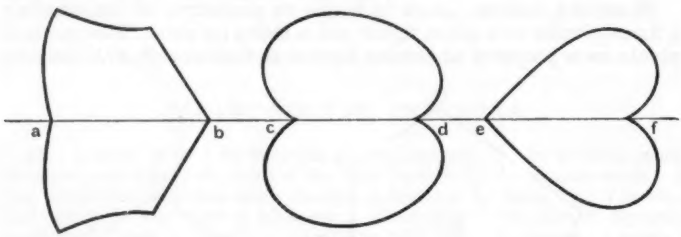


FIG. 4.

The original problem of Fig. 1 is a case of this sort. To divide the area into three equal parts we proceed as follows. Divide the median AB into three equal parts, $A1$, 12, $2B$ (Fig. 5). On each of the bases $A1$, $A2$ draw figures similar to the upper half of the given figure, working from the point A. Corresponding to these, draw figures similar to the lower half of the given figure on bases $B1$, $B2$, working from B. Then the complete segments, as shown shaded, will be each one-third of the given area.

The general proof is as follows : in Fig. 6 let the point C divide AB into two portions x and y. Considering the upper similar area a on base x, we have

$$\text{area } a : K = x^2 : (x+y)^2.$$

Similarly, $\quad \text{area } a_2 : K = y^2 : (x+y)^2.$

Hence $\quad \text{area } a_1 = K\left\{1 - \dfrac{y^2}{(x+y)^2}\right\},$

and $\quad \text{area } (a + a_1) = K \cdot \left\{\dfrac{x^2}{(x+y)^2} + 1 - \dfrac{y^2}{(x+y)^2}\right\}$

$$= 2K \cdot \frac{x}{x+y} = \frac{x}{l} \cdot 2K.$$

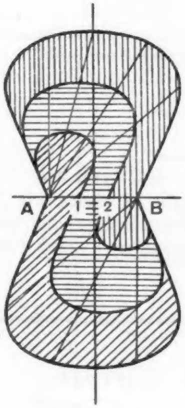


Fig. 5.

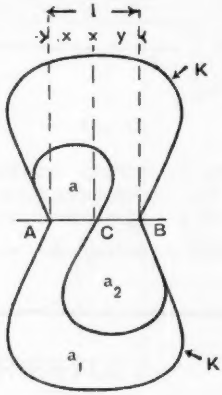


Fig. 6.

This is the result. But there is more; if the upper and lower contours of the given figure are equal, we have (Fig. 7)

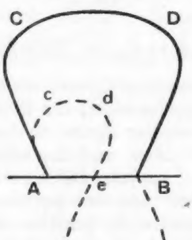


Fig. 7.

contour $Acde$: contour $ACDB = Ae : AB$,

contour $efgB$: contour $ACDB = eB : AB$.

Adding, $\quad$ contour $AcdefgB$ = contour $ACDB$.

This property is interesting in the study of waves. For instance, a wave (Fig. 8a), crossing the base line three times, would yield a similarly derived wave *of equal length*, crossing the base line five times (Fig. 8b), and this a wave crossing nine times (Fig. 8c), and so on, until we get as small a wave as we please at the origin 1; we might consider this small wave as a terminus of propagation.

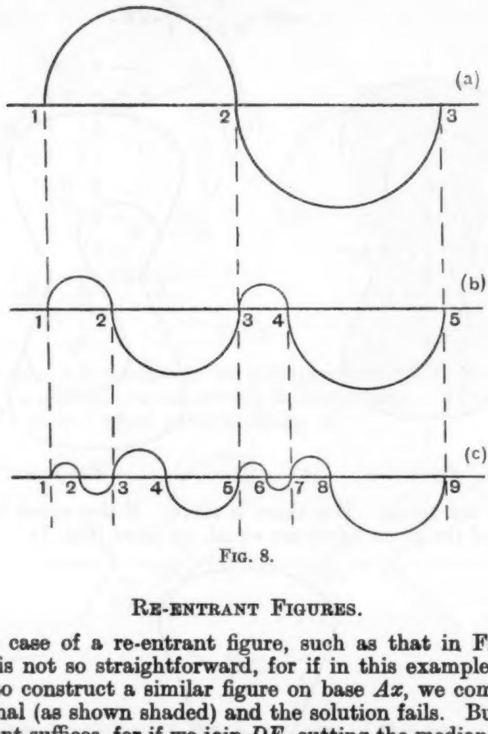


Fig. 8.

Re-entrant Figures.

In the case of a re-entrant figure, such as that in Fig. 9, the solution is not so straightforward, for if in this example we work from A to construct a similar figure on base Ax, we come outside the original (as shown shaded) and the solution fails. But a slight adjustment suffices, for if we join DF, cutting the median AB at O, the given figure is divided into two portions ADF, DBF, with AB as a common median, and each portion can be divided into three equal parts separately. A caution is necessary: the contour $AabcdeB$ is *not* in this case equal to the contour $ADEB$ but to the contour $ADOEB$, that is, $ADEB+2DO$, or to contour $ADGB$.

Other Problems.

If we cause a body to spin round its median, it will readily be seen that, working along the same lines, we can establish the relationship between the various surfaces of revolution of its similar

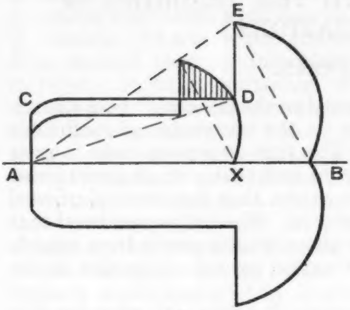


FIG. 9.

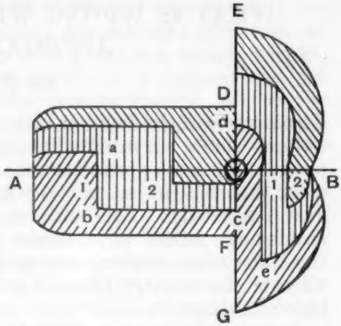


FIG. 10.

figures. Another line of thought concerns centres of gravity. Indeed, the study of similar figures is most interesting, and might be of help in understanding problems connected with the development of " form " in nature.

A. VAN DER MEERSCH.

HENRY JAMES PRIESTLEY.

BY the death, on 26th February last, of Professor H. J. Priestley, after a long illness, the Queensland Branch of the M.A. has lost its foundation President and the Association a distinguished member.

The late Professor was born in April 1883, and was educated at Mill Hill School and Jesus College, Cambridge; he was Fifth Wrangler in the Mathematical Tripos of 1905, and was placed in the second division of the First Class of Part II of the Tripos in 1906. He pursued investigations at Cambridge in Experimental Physics during 1906 and 1907, and his paper on the Diffraction of Electromagnetic Waves was commended by the Smith's Prize examiners. He took his M.A. degree in 1909, and before coming to Queensland in 1911 he was Lecturer in Mathematics in the University of Manchester. He was one of the original group of Professors appointed at the foundation of the University of Queensland, being the first Professor of Mathematics and Physics.

There was no department of University life in which he did not take part, and his influence was felt everywhere. He was an enthusiastic teacher, a man most likeable, of consummate tact and of confirmed modesty. He has left behind him the impression of a most charming personality.

J. P. MCCARTHY.

WHAT IS WRONG WITH THE TEACHING OF APPROXIMATIONS?

By S. Inman.

The reports of the London University on the General School Examinations throw an interesting light on the knowledge of candidates of the subject of approximations. The 1928-30 reports make it quite clear that the subject is badly taught and that glaring errors occur very frequently. The 1929 report states that this fault is general and is not confined to particular schools. When it is considered that the University examines annually about 25,000 pupils from schools all over the country there is justification for the suggestion in the title of this paper.

Some teachers recognising the state of affairs ask whether it is worth while teaching the subject. Scientists, and all who have to deal with measurements—directly or indirectly—know that their data are approximate and they must value any deduced results accordingly. A pure number rarely occurs by itself. It is nearly always associated with a concrete quantity. As a result, practically all values are approximate. That being the case, it is certainly worth while dealing with the subject, not only from the educational and practical points of view, but also for the mathematical ideas gained.

It is as well to divide the subject into two parts, approximations and contracted work. Contracted work consists of multiplications and divisions entirely disconnected from any concrete quantity. Pupils are set to work these by the score, and the only interest in this work—from the teacher's point of view—is observing how quickly the pupil forgets the rules, and how rarely he applies them except under compulsion.

"Approximations", on the other hand, consists in assessing the accuracy of a measurement and considering how it affects an answer when used in any process such as multiplication. To be brief, it is possible to have valuable ideas on approximation without having heard of contracted work. On the other hand, it is possible for a pupil to know how to contract rapidly without having a single idea of value.

As an illustration, suppose it is required to find the area of a rectangle a measurement of which gives the sides as 4·67 in. and 3·85 in. In the first place it should be realised that no actual rectangle has those measurements. All that we can say is that, if our ruler is accurate, 4·67 in. and 3·85 in. are the measurements to the nearest ·01 in. The question of what is a reasonable answer for the area must be considered. This will lead to a discussion of the terms absolute and relative errors, significant figures, and three-figure accuracy, and the pupil will be led to see that any figure beyond the third is not merely valueless but is wrong. If the pupil can multiply 4·67 by 3·85, obtaining 17·9795, and give his answer as 18·0 sq. in., he has acquired a valuable idea. Whether he does or does not contract the multiplication is relatively unimportant, and

in most cases the labour that can be saved is insignificant. The process of contracted work may be compared with the Italian method of division. This is certainly labour saving, yet it is seldom taught. It is realised that the important thing is, do a division and get it right. Further shortening of the process can be left to the specialist. An answer should not contain more significant figures than the least accurate of the data used. There are some who say that as the last figure cannot always be relied upon, the answer should contain fewer significant figures, even two figures less being sometimes recommended. There are often grave disadvantages in these recommendations. It is advocated here that the answer should contain the same number of significant figures as—not fewer than—the least accurate of the data used. It is true that the last figure is sometimes slightly in error. This should be admitted and pointed out. Nevertheless, the answer is more accurate than that obtained by curtailing the number of figures. Moreover, it is usually possible to assess the error. Thus, a chemist in making a weighing and obtaining 3·074 grms., knows that there may be a possible error of $\pm\cdot 002$ grm. Which is the better value—3·07 grm. or 3·074 grm. ? The latter obviously gives a better idea of the weight although we cannot rely on the last figure being correct. If, on the other hand, we say that as we cannot rely on the last figure we must reject it, we find that, pursuing the argument, we may be obliged to reject still more figures. Thus if the error is $+\cdot 002$ grm., the weight is nearer 3·08 grm. Hence we cannot say for certain whether the weight is nearer 3·07 or 3·08 grm. and the only weight we can be certain about is 3·1 grm., which is an absurd value for an accurate weighing. The chemist would be scornful of such arguments and stick to 3·074 grms. As he deals with actual things, his views should command greater respect. The best statement of the weight is of course $3\cdot 074 \pm \cdot 002$ grm., because that tells us all the information about the weight. As a further illustration of the absurdity of being meticulous, consider what value should be given to 6·5 in. correct to the nearest tenth of an inch. It is not possible to say whether the length is nearer 6 or 7 in., in which case it is impossible to state any answer. Again $6\cdot 5 \pm \cdot 05$ in. tells us everything.

More use should be made of the notation $\pm$. A simple illustration of its use is to find the sum of the three measured lengths 23·4 in., 16·7 in., and 43·2 in., each being correct to 0·1 in. The working should be set out as follows :

$$\begin{array}{l} 23\cdot 4 \pm \cdot 05 \text{ in.} \\ 16\cdot 7 \pm \cdot 05 \ \ ,, \\ 43\cdot 2 \pm \cdot 05 \ \ ,, \\ \hline 83\cdot 3 \pm \cdot 15 \text{ in.} \\ \hline \end{array}$$

Although there is a doubt about the last figure in 83·3 in., that is a better answer than 83 in., and $83\cdot 3 \pm \cdot 15$ in. is the best answer.

If we add n numbers each correct to the nearest unit, G. W. Palmer states that it is so highly improbable as to be practically out of the question that the error is greater than $\sqrt{n}$. Thus we

are quite safe in adding 25 sums of money expressed to four places of decimals, because the error would be less than $5 \times £\cdot 0001$, *i.e.* less than half a farthing. This is a striking instance where the exact attitude of mind is out of contact with reality.

It should be noted that even allowing for the maximum error we can rely on the same number of significant figures in our answer as in the individual measurements. Thus if we added ten lengths each 23·4 in., the total is $(23\cdot4 \pm \cdot05 \text{ in.}) \times 10 = 234 \pm \cdot5$ in., which is correct to the same number of significant figures as 23·4 in. Actually the error would be much less than ·5 in., so that the more lengths added, the more significant figures we can rely upon in our answer—not less as might be imagined. Thus if we added 100 lengths each of three significant figures and correct to the nearest tenth of an inch, the total would be certainly correct to four significant figures.

Returning to the rectangle whose sides are 4·67 in. and 3·85 in. to $\frac{1}{100}$ in. we shall consider the error more in detail. Although the absolute error is sometimes taken into account, the relative error is in nearly every case the more important. Furthermore, it involves the idea of a ratio, which is the most important idea in elementary mathematics. Use of the relative error will help considerably in estimating the value of an approximation. The maximum relative errors in the measurements 4·67 in. and 3·85 in. are approximately

$$\frac{\frac{1}{2}}{450} \text{ or } \frac{1}{900} \text{ and } \frac{\frac{1}{2}}{400} \text{ or } \frac{1}{800},$$

$$4\cdot67 \times 3\cdot85 = 17\cdot9795,$$

$$\frac{1}{900} \text{ of product} = \cdot02,$$

$$\frac{1}{800} \quad ,, \quad = \cdot02.$$

The maximum error is therefore $\pm\cdot04$ sq. in. We are therefore justified in calling the area 18·0 sq. in., although there is a slight possibility of the answer being nearer 17·9 sq. in. The best answer is $17\cdot98 \pm \cdot04$ sq. in. as it gives all the information we can desire. To call the answer 18 sq. in. may be more precise, but is much less accurate.

Here is another illustration from mensuration. If 3·14 is taken for π, how many significant figures should the answer contain? I say three, and admit the possibility of a slight error in the last figure. But even that does not prevent me obtaining quite accurate information about the third figure. Thus with a circle of diameter 20·63 in., taking π as 3·14, multiplication gives 64·7782. Taking π as 3·1416, the correction, or relative error, is $+\frac{1\frac{1}{2}}{3000}$ or $+\frac{1}{2000}$, which gives $+\cdot03$ in. Hence the corrected answer is 64·81 in. This is therefore correct not merely to three but actually four significant figures, although π was taken to three significant figures. In this case it is known that the error is plus, not $\pm$, and hence the error can be corrected with certainty. Thus if π is taken as

3·14, the answer should be given to three significant figures—not less. We admit the possibility of a slight error in the last figure, but at the same time we can soon obtain the error. If we do not always insist on the correction being made, it is simply for convenience in classwork.

In most cases it is sufficient to give the relative error to one significant figure, so that this can be obtained mentally and rapidly. A correction is then soon obtained, not merely for the third significant figure (in this illustration), but which actually enables us to get the next significant figure.

Useful work could be done with examples involving correction errors. I shall indicate two other examples which might be set.

Example 1. The sides of a rectangle are 2·68 in. and 4·12 in. to the nearest ·01 inch. Multiplication gives 10·04 sq. in. What is the maximum error in the answer ?

Method. Relative errors are $\frac{1}{500}$ and $\frac{1}{800}$.

$$\begin{aligned} \tfrac{1}{500} \text{ of product} &= \cdot 02 \text{ sq. in.} \\ \tfrac{1}{800} \quad\;\; \text{,,} \quad &= \cdot 01 \;\;\text{,,} \\ \hline \text{Maximum error} &= \pm\cdot 03 \text{ sq. in.} \end{aligned}$$

Example 2. In multiplying 34·0632 by 5·26323, the numbers are shortened to 34·06 and 5·263, and the product obtained is 179·25. Make a correction.

Method. Relative errors in numbers are

$$\frac{1}{10{,}000} \text{ and } \frac{1}{20{,}000}.$$

$$\begin{aligned} & & 179\cdot 25 \\ \text{Correction for 1st number} &= +\frac{1}{10{,}000}\text{th} = + & 2 \\ \text{,,} \qquad \text{2nd number} &= +\frac{1}{20{,}000}\text{th} = + & 1 \\ \hline \text{Corrected product} & & 179\cdot 28 \end{aligned}$$

Having established the rule that an answer should not contain more figures than the least accurate of the data, the teacher should give practice with examples based on measurements, and answers should be required to a reasonable degree of accuracy. The pupil must be capable of deciding what constitutes a reasonable number of figures. Thus, if asked to find the circumference of a circle to four significant figures, he should know, without asking, that 3·142 is the least accurate value he should take for π. In working an example on numerical trigonometry he should realise the absurdity of giving more than four significant figures in the answer if he is working from four-figure tables.

Cases will occur where the number of figures in our data is much more than is required in our answer. Thus in 1912 the output of

coal in the British Isles was given as 264,595,395 tons, and the persons employed as 1,072,393. What is the output per man to the nearest ton ? The rough estimate is 260. Here the problem of contraction will arise. The pupil will have no difficulty in seeing that if we retain the same number of significant figures—three in this case—in both dividend and divisor, our answer is practically correct except for a slight doubt about the last figure. Hence work with an extra figure in these numbers. This is the only rule which need be given for any single process. It is simple, soon learned, and can be immediately applied. The above example would be worked as follows :

$$\frac{264{,}595{,}395}{1{,}072{,}393}=\frac{264{,}600{,}000}{1{,}072{,}000}=\frac{264{\cdot}6}{1{\cdot}072}=246{\cdot}8$$

$$=247 \text{ approx.}$$

No further rules are given, because the example could be worked by ordinary division, and much labour is saved. The rule should now be applied to other problems. However, the examples should refer to data and measurements, and should not be abstract multiplications and divisions. Most text-books contain examples of the type quoted above.

I will now consider what error is likely in working the above rule. Considering the example taken,

$$\begin{aligned} &\text{Quotient} = && 246{\cdot}8\\ \text{Correction for dividend} &= -\frac{1}{50{,}000}\text{th} = - && {\cdot}005\\ \text{,,}\qquad \text{divisor} &= -\frac{1}{2{,}500}\text{th} = - && {\cdot}1 \end{aligned}$$

These errors do not affect the units figure. If necessary, they can be used to improve the result. Thus the answer to four significant figures is 246·7.

Again, if $46{\cdot}48 \times 3{\cdot}573$ is required to two significant figures,

$$\begin{aligned} 46{\cdot}5 \times 3{\cdot}57 &= 165{\cdot}99\\ \text{Corrections}\quad - &\quad {\cdot}08\\ + &\quad {\cdot}16 \end{aligned}$$

Again the errors are insignificant. The maximum error might occur when the leading figures, here 4 and 3, do not give double figures when multiplied and give a product as high as 8 or 9. In such a case, if the answer is required to the nearest unit, the error may be as high as ·2 or ·3, or even more. The magnitude of this, however, is more apparent than real. It should be realised that giving an answer to three significant figures is an elastic requirement. Thus, suppose we take 872 for 872·7 and 166 for 166·4, although the former answer might not be considered correct to three significant figures, it is a better answer than the latter, as the relative error is only one-third that of the latter. Hence, although with certain leading

figures there is sometimes a higher absolute error, there is an automatic compensation because the relative error drops correspondingly. We can therefore always rely on the method of using an extra figure to give a satisfactory result.

It will be noticed that I have so far not recommended the contraction of the partial products. In the first place we have a single rule which can immediately be put to use to effect a drastic reduction in the work, and in the great majority of cases there is little or no saving of work in applying additional rules. In fact, usually a lot of time is wasted in thinking out what to do to find that perhaps only one or two figures have been saved. Secondly, and this is important, contraction of the partial products is really a refinement and of subsidiary importance.

Examples could be done involving two or more processes. The same rule is applied. Thus with an example of the type $(A \times B)/C$ to three significant figures, use only four figures of A and B. When the product has been obtained, cross out the figures after the fourth, and divide by C, which has been shortened likewise.

It is interesting to refer to the syllabuses of the examinations set by the Civil Service Commission. Those who have any acquaintance with the questions set will agree that for excellence and testing of common sense they are unsurpassed by those of any other examining body. Having for their aim the testing of intelligence, especially with situations which might arise in ordinary life, the questions bear the closest resemblance to problems of everyday experience. Any opinion, therefore, expressed by the Commission is well worthy of consideration. I have read the syllabuses of several of their examinations, and in most of them it reads : " Contracted methods are not required. Candidates may work with all the figures supplied or make legitimate discards at any stage. An arithmetical result may be asked to a certain degree of approximation, or the data may themselves be only approximate. In such a case to give the result to a greater degree of accuracy than is asked for, or justified by the data, will entail loss of marks ". In my opinion this is a very sane view of the matter.

I come now to the question of further contraction. When the class is ripe for additional rules, the work might arise naturally from the type of examples they have already done, particularly one involving several processes. Or a single process may be selected where there might be much saving of work.

This brings us to the problem of what rules should be adopted. With regard to the present rules, I must say at the outset that they must be scrapped. There are two big objections. One is that they resemble a man digging up a kitchen garden with a kitchen fork. If we wished to weigh a couple of pounds of potatoes, would we use a chemical balance ; or if we required a length to the nearest inch, would we measure with the vernier callipers which give the answer to the hundredth of an inch ? Yet that is exactly what is generally done. The tailor measures to the nearest half-inch and makes a perfect coat. He deals with actual things and knows—

perhaps unconsciously—the value of the last figure in an approximation. Why do so many teachers insist on two extra figures in the working? If we want an answer to the nearest unit, and the value before approximating is near 246·5, does it really matter whether we take 246 or 247? If the correct answer is a trifle nearer the latter, is that answer so much a better one? It is also wrong, and is about as much out as the other answer. For all practical purposes, both answers are equally good. It seems to me that when we go searching among the second decimal figures we lose sight of the meaning of the term "approximation". I worked through fifty to sixty examples on multiplication. They were not specially selected, but taken at random from a text-book. Working to one extra figure in the partial products, the same answer was obtained as with two extra figures except in two cases. In these, working with two extra figures gave the first decimal place as 5 and 4 respectively. I submit that in these cases it did not matter which of the two answers were given. If it was an important matter, then the answer should have been demanded to the nearest tenth, instead of nearest unit. Once again let us think of actual things, not abstract multiplications, which have led us astray too long. This working to two extra figures arises out of the desire of some teachers to know definitely which is the nearer of two answers. They think that they have obtained the solution, but I must undeceive them, for it is impossible to decide the question always by contracted work. Thus with the sum $6323{\cdot}451 \times 4{\cdot}868785$ to five significant figures, *i.e.* to the nearest unit. Working with two extra figures we obtain

$$
\begin{array}{r@{}l}
6323{\cdot}451 & \\
4{\cdot} & 868785 \\
\hline
25293{\cdot}80 & \\
5058{\cdot}76 & \\
379{\cdot}40 & \\
50{\cdot}58 & \\
4{\cdot}42 & \\
{\cdot}50 & \\
3 & \\
\hline
30787{\cdot}49 & \quad \text{or 30787 to the nearest unit.}
\end{array}
$$

If the work is not contracted, the answer is 30787·52 ..., *i.e.* 30788 to the nearest unit. Thus working to two extra figures does not enable us to say with certainty which answer is nearer. The discrepancy in this example is admittedly exceptional, but obviously whenever the last two figures are ·49, ·50, or ·51, *i.e.* in 3 per cent. of cases, there must be a doubt as to which answer is better. It is no use working to three extra figures—one book recommends this—because we still could not always be certain, and never can be. This, I hope, explodes the theory of working to two extra figures.

Perhaps the most serious charge I make against the usual rules is that they are unworkable. They break down in all but the simplest cases. That is why a boy, when he thinks you are not looking, will

multiply without contracting in spite of months of energetic teaching. He cannot be bothered to think out complicated rules, and knows that he can get the result more quickly and without having to rack his brains. Apply the rules to work out an example of the type $A/(B \times C)$, and we have to spend half our time in thinking out what to do. Most books recognising these difficulties, ignore any examples of more than a single process.

The value of a number depends on the number of its significant figures, not its decimal places. The rules of contraction are at present based mainly on decimal places. That is why they are unworkable.

The remedy is obvious. Everything must be thought of in terms of significant figures, never in decimal places. To summarise the rules briefly, no matter how many processes there are, no intermediate step should have more than two extra significant figures (including the extra figures in the working) in its answer. Justification for this is easily obtained by a consideration of the relative error.

We must first, however, consider what to do in the case of a single process. It is important to remember that what we are doing now is a refinement of a more important rule. No new rule, therefore, must in any way confuse the previous rule. If there is any such danger, we must not go any farther because, not only will we be wasting time, but we shall undo the previous good work. Fortunately it is possible to contract still further with quite satisfactory accuracy by merely adding something to the rule already learned.

Thus with $48{\cdot}628 \times 12{\cdot}932$ to three significant figures, according to the rule already given, we must multiply 48·63 by 12·93. The pupil can now, if he desires, finish by ordinary multiplication. Further contraction does not mean scrapping this rule. The pupil merely dispenses with certain figures. The working is as follows, details being left to the reader :

$$
\begin{array}{r@{}l}
486{\cdot}3 & \\
1{\cdot}293 & \\
\hline
486{\cdot}3 & \\
97{\cdot}3 & \\
43{\cdot}7 & \\
1{\cdot}4 & \\
\hline
\underline{628{\cdot}7} & \quad \text{or 629 to three significant figures.}
\end{array}
$$

Of course it is essential to add the carrying figure from the last figure crossed out.

Similarly with division. One could take, as a standard illustration, an example with too many figures in the divisor. Thus :

$$570{\cdot}0423 \div 24{\cdot}34728 \text{ to four significant figures.}$$

Cross out the figures beyond the fifth and we obtain $570{\cdot}04 \div 24{\cdot}347$. Again the pupil can now finish by ordinary division, or can contract

further. In this standard illustration we decide not to bring down any figure from the dividend. The working is :

$$
\begin{array}{r|r|}
\multicolumn{1}{r}{} & \multicolumn{1}{r}{234{\cdot}1} \\
\cline{2-2}
2{\cdot}4{,}3{,}4{,}7\,) & 570{\cdot}04 \\
 & 486{\cdot}94 \\
\cline{2-2}
 & 83\ 10 \\
 & 73\ 04 \\
\cline{2-2}
 & 10\ 06 \\
 & 9\ 74 \\
\cline{2-2}
 & 32
\end{array}
$$

Again the carrying figure must be added. Modifications of this standard type can be discussed. Thus if the divisor is one short of the standard number, contraction must be deferred for one step.

With an example of the type $(A \times B)/(C \times D)$, to four significant figures, work the numerator and denominator so that the products do not contain more than six figures (including extra figures), and work the final division with five figures in the divisor.

I will conclude by giving further applications of the use of the relative error from the point of view of shortening our work.

Example 1. $62{\cdot}34 \times {\cdot}4996$ to four significant figures.

$$
\begin{array}{lr}
62{\cdot}34 \times {\cdot}5 \qquad = & 31{\cdot}17 \\
\text{Correction } -\tfrac{1}{1000}\text{th} = - & 3 \\
\text{Corrected product} & \underline{31{\cdot}14}
\end{array}
$$

Example 2. This is slightly longer, but is a type of more frequent occurrence.

$54{\cdot}38 \times 4{\cdot}205$ to four significant figures.

$$
\begin{array}{lr}
54{\cdot}38 \times 4{\cdot}2 \text{ soon gives} & 227{\cdot}40 \\
\text{Correction } +\tfrac{1}{800}\text{th} \quad = + & 3 \\
\text{Corrected product} & \underline{227{\cdot}7}
\end{array}
$$

This is obviously correct to four figures. Sometimes both numbers can be shortened.

Example 3. $54{\cdot}02 \times 4{\cdot}205$.

$$
\begin{array}{lr}
54 \times 4{\cdot}2 \text{ soon gives} & 226{\cdot}8 \\
\text{for } 54{\cdot}02 \text{ correction } \tfrac{1}{3000}\text{th } + & {\cdot}1 \\
\text{,, } 42{\cdot}05 \quad \text{,,} \quad \tfrac{1}{800}\text{th } + & {\cdot}3 \\
\text{Corrected product} & \underline{227{\cdot}2}
\end{array}
$$

Example 4. $7{\cdot}52 \times 2{\cdot}53$.

$$2{\cdot}53 \doteqdot 2{\cdot}5 = \tfrac{10}{4}.$$

$$
\begin{array}{lr}
7{\cdot}52 \times 2{\cdot}5 & = 18{\cdot}8 \\
\text{Correction } \tfrac{1}{80}\text{th} & + \ {\cdot}2 \\
\text{Product to three significant figures} & \underline{19{\cdot}0}
\end{array}
$$

Example 5. $21 \cdot 37 \times \pi$ to five significant figures.

Taking π to four figures (3·142),	
Product is	67·176
Correction ($-\frac{1}{8000}$th) $-$	8
Corrected value	67·168

The method of relative error may be used effectively to obtain more drastic contraction with any example of multiplication and division. Thus taking any example at random :

Example 6. $34 \cdot 18 \times 2 \cdot 683$ to three significant figures.

Taking only THREE figures of each,		
$34 \cdot 2 \times 2 \cdot 68$	=	91·66
For 34·18 correction ($\frac{1}{1700}$th)	= $-$	5
„ 2·683 „ ($\frac{1}{900}$th)	= +	10
		91·71

which gives the answer not merely to three but even four figures, although we started with only three figures from each number.

Finally, I should like to add a reminder that approximations is not a subject which should be kept in a water-tight compartment and used only in the special lessons on the subject. I have pointed out that the subject should not be kept on a theoretical plane, but should be taught in connection with things and measurements. The science master, who frequently has recourse to measurements, has excellent opportunities for driving home ideas on approximation, but opportunities are not wanting in the mathematics lessons. Every time a mensuration example is dealt with, every time a value for π has to be decided upon, every time we use a trigonometrical table or a table of logarithms, we are dealing with approximations.

One more word to the believer in the old rules. If you insist on your rules, be consistent and carry them out. Whenever you use 3·142 for π in a mensuration sum you must never give the answer to more than two significant figures. If you use $3\frac{1}{7}$, which you probably prefer, one significant figure is the limit I allow you, and whenever you use logarithmic or trigonometrical tables, I will let you off with two significant figures with a caution.

Finally, to sum up in four words, " Think in significant figures ".

S. I.

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., Derby School, Derby, to whom all inquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the text-books on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. The names of those sending the questions will not be published.

ON THE AREA OF A CIRCLE.

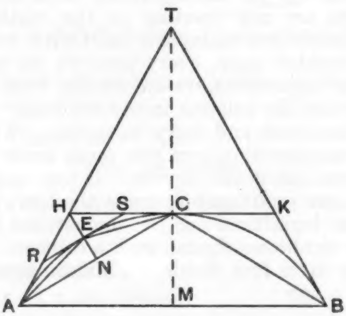
By C. H. Hardingham.

At the beginning of the seventeenth century π had already been calculated to 35 places of decimals by the Archimedean method of finding in succession the perimeters of inscribed and circumscribed polygons, each having twice as many sides as its predecessor.

But at this time several mathematicians, notably Snell, were seeking shorter methods of approximation, and finally Huygens in thirteen propositions proved or disproved the assumptions they made, and in three more propositions, making use of the known position of the centre of gravity of a parabolic segment and a theorem of his own on the centre of gravity of a segment of a circle, showed how to find a closer upper limit for π from any polygon than had yet been obtained, and by a seventeenth theorem " on centre of gravity ", of which he does not give the proof or even the enunciation, he found also a closer lower limit.

It is perhaps curious that none of the mathematicians of the time who were looking for such things hit upon the following simple theorem which leads to closer limits than those found by Huygens :

1. AB is a chord of a circle. C bisects the arc AB. The tangent HCK, parallel to AB, meets the tangents at A and B in H and K. E bisects the arc AC, and the tangent RES, parallel to AC, meets HA, HC in R and S. HE bisects AC at right angles at N.



If AB is a side of a regular inscribed polygon of n sides, and A_n denotes the area of such a polygon, A_n' that of the corresponding circumscribed polygon, then

$$A_{2n} - A_n = n \,.\, \triangle ACB,$$
$$A_{4n} - A_{2n} = 2n \,.\, \triangle AEC,$$
$$A_{2n}' - A_{4n}' = 2n \,.\, \triangle HRS.$$

Now
$$\triangle HRS : \triangle AHC = HE^2 : HN^2,$$
$$\triangle AEC : \triangle AHC = EN : HN,$$

$$\triangle ACB : \triangle AHC = CA^2 : CH^2$$
$$= 4CN^2 : CH^2$$
$$= 4EN^2 : HE^2,$$

since CE bisects the angle HCN. So

$$\triangle HRS \times \triangle ACB = 4(\triangle AEC)^2,$$

but $$\triangle HRS(\triangle AEC - \triangle HRS) < \tfrac{1}{4}(\triangle AEC)^2,$$

as may be seen by taking lengths proportional to the triangles and using Euclid II. 5.

Thus $$\triangle AEC - \triangle HRS < \tfrac{1}{16}\triangle ABC.$$

Accordingly $8(A_{4n} - A_{2n}) - 8(A_{2n}' - A_{4n}') < A_{2n} - A_n$,

or $8A_{4n} + 8A_{4n}' - A_{2n} < 8A_{2n} + 8A_{2n}' - A_n$.

Similarly $8A_{8n} + 8A_{8n}' - A_{4n} < 8A_{4n} + 8A_{4n}' - A_{2n}$,

and proceeding thus we have

$$8A_{2m} + 8A_{2m}' - A_m < 8A_{4n} + 8A_{4n}' - A_{2n},$$

where m is equal to n multiplied by as high a power of 2 as we please, so that we can make each of the areas A_{2m}, A_{2m}', A_m differ from A, the area of the circle, by as little as we please, from which it is clear that

$$15A < 8A_{2n} + 8A_{2n}' - A_n,$$

or taking the radius as unity,

$$\pi < \tfrac{1}{15}(8A_{2n} + 8A_{2n}' - A_n).$$

When $n=2$, we have $\pi < (8 \times 6 - 0)/15 = 3{\cdot}2$, since $A_2 = 0$, $A_4 = 2$, $A_4' = 4$.

For $n=3$, $\pi < 3{\cdot}146 \ldots$; $n=4$, $\pi < 3{\cdot}142 \ldots$; $n=6$, $\pi < 3{\cdot}1416 \ldots$.

Huygens calculated from the sides of the polygons, but his formula expressed in terms of areas is

$$\pi < A_{2n} + \tfrac{1}{3}(A_{2n} - A_n)(4A_{2n} + A_n)/(2A_{2n} + 3A_n),$$

which gives for $n = 2, 3, 4, 6$, $\pi < 3{\cdot}33 \ldots$, $3{\cdot}154 \ldots$, $3{\cdot}1438 \ldots$, $3{\cdot}1418 \ldots$.

If in the new formula A_{2n}' is replaced by $2A_{2n}A_n'/(A_{2n} + A_n')$ or by

$$\frac{A_{2n} + A_n'}{2} - \frac{(A_n' - A_{2n})^2}{2(A_{2n} + A_n')},$$

it appears that the amount of calculation required by the two formulae is much the same, since A_n' or its equivalent has to be calculated in order to obtain the value of A_{2n}.

2. A lower limit for π and a closer upper limit may be found from the following observations:

2·1. $$\triangle HRS : \triangle EAC = HS : CN,$$

and $$\tfrac{1}{4}CN^2 - HS(CN - HS) = \tfrac{1}{4}(2HS - CN)^2,$$

whence $$\tfrac{1}{16}\triangle ABC - \triangle AEC + \triangle HRS = \tfrac{1}{4}\triangle HRS\left(\frac{2HS - CN}{HS}\right)^2.$$

If now we take AB as a side of a regular inscribed polygon of $2n$ sides and use $f(2n)$ to denote $\frac{1}{15}(8A_{2n}+8A_{2n}'-A_n)$, we have

$$f(4n)-f(8n)=\frac{8n}{15}\,\triangle HRS\,.\left(\frac{2HS-CN}{HS}\right)^2,$$

and similarly if the tangents AH, BK meet at T, and TC meets AB at M,

$$f(2n)-f(4n)=\frac{4n}{15}\,\triangle THK\left(\frac{2TH-AM}{TH}\right)^2.$$

2·2. $$\tfrac{1}{4}\triangle HAC<\triangle HRS<\tfrac{1}{4}\triangle HAC\,.\frac{CA}{AM},$$

so $$\tfrac{1}{8}\triangle THK\,.\frac{AM}{AT}<\triangle HRS<\tfrac{1}{8}\triangle THK\,.\frac{AC}{AT}.$$

2·3. $$\frac{AM}{AT}=\frac{CM}{CT}=\frac{AM-CH}{CH}=\frac{AM}{AC}\cdot\frac{AC}{CH}-1=2\frac{CN^2}{CH^2}-1,$$

so $$\frac{AT-AM}{AT}=\frac{2CH^2-2CN^2}{CH^2}=\frac{2(CH-CN)(CH+CN)}{CH^2}$$

$$=\frac{2(CH-CN)}{CH}\cdot\frac{HS+SE}{SH}=\frac{2(CH-CN)}{SH},$$

whence $$\frac{CH-CN}{AT-AM}=\frac{SH}{2AT}$$

and $$\frac{HS-SE}{TH-HC}=\frac{SH^2}{2TH\,.\,CH}.$$

2·4. $$2HS-CN=(HS-SE)+(HC-CN)$$

and $$2TH-AM=(TH-HC)+(TA-AM),$$

so $$\frac{TH}{2AT}>\frac{2HS-CN}{HS}:\frac{2TH-AM}{TH}>\frac{SH}{2CH}>\frac{1}{4}.$$

Thus we have

$$\frac{f(4n)-f(8n)}{f(2n)-f(4n)}>\frac{1}{4}\cdot\frac{HA}{HT}\cdot\frac{1}{16}=\frac{1}{64}\cdot\frac{HA}{HT},$$

$$\frac{f(4n)-f(8n)}{f(2n)-f(4n)}<\frac{1}{4}\frac{AC}{AT}\cdot\frac{TH^2}{4AT^2}<\frac{1}{16}\cdot\frac{AC}{AT}\cdot\frac{TH^2}{4TH\,.\,AH}$$

$$=\frac{1}{64}\frac{AC}{AT}\cdot\frac{TH}{AH}=\frac{1}{64}\frac{AC}{AT}\cdot\frac{AT}{AM}$$

$$=\frac{1}{64}\cdot\frac{AC}{AM}.$$

2·5. Finally, if d_4 denotes $f(2n)-f(4n)$ and we take the radius of the circle as unity, so that its area is π, we have

$$f(4n)-\pi=d_8+d_{16}+d_{32}+\dots,$$

and if all the ratios d_8/d_4, d_{16}/d_8, ... are less than r^{-1}, this sum is less than $$d_4(r^{-1}+r^{-2}+r^{-3}+\dots)=d_4/(r-1);$$

similarly, if all the ratios are greater than r^{-1}, the sum is greater than $d_4/(r-1)$. Noticing that

$$AM/AC=A_{4n}/A_{8n},\quad HT/HA=A_{4n}/A_{2n},$$

and $$\frac{A_{4n}}{A_{8n}} > \frac{1}{2}\left(1+\frac{A_{2n}}{A_{4n}}\right),$$
we see that
$$\pi > f(4n) - \{f(2n) - f(4n)\} \Big/ \left\{32\frac{A_{2n}}{A_{4n}} + 31\right\};$$
$$\pi < f(4n) - \{f(2n) - f(4n)\} \Big/ \left\{64\frac{A_{4n}}{A_{2n}} - 1\right\}.$$
Thus we have
$$n=3 : 3{\cdot}141586 < \pi < 3{\cdot}141603;$$
$$n=4 : 3{\cdot}141591 < \pi < 3{\cdot}141594.$$
Huygens' limits were
$$n=3 : 3{\cdot}14135 < \pi < 3{\cdot}14181;$$
$$n=4 : 3{\cdot}141554 < \pi < 3{\cdot}141627.$$
The true value is of course 3·1415926 ... .

If it is preferred, the series in 2·5 may be summed geometrically, and, on the other hand, if the ratio d_8/d_4 is simplified algebraically, closer limits of the type given may be found for it.

C. H. HARDINGHAM.

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880. At a considerably higher level we find the contemporary mathematician who has still to learn the real meaning of "experimental verification", and who is habituated to treating the schemes of concepts in his brain as truer than fact, at odds with the modern biologist. Still constrained in the logical net, he shakes his head at the "unphilosophical" ease of the latter's mental movements. He objects to conclusions that are not final and exactly proved. He has not learnt to rest in a provisional conclusion, and clings to the delusion that purely symbolic processes can win truth from the unknown. His symbolic processes never do win truth from the unknown, but he fancies that they justify an attitude of disapproval towards the pragmatical acceptances of practical science. But in the long run perhaps even the mathematicians will become scientific.—H. G. Wells, *The Work, Wealth and Happiness of Mankind*, p. 69 (1932). [Per Prof. E. H. Neville.]

881. Did you know that the Polish Government pays the travelling expenses of *mathematicians* to go from one place to another whenever there is a meeting? But the Government does not do this for the other sciences.—[R. C. A., in a letter to E. H. N., Feb. 24, 1932.]

882. HEAVEN. "The one idea . . . that has seemed to me to overshadow all the rest, is that of *Eternity*—involving, as it seems to do, the necessary *exhaustion* of all subjects of human interest. . . . Take the subject of . . . curves of the second degree. In a future Life, it would only be a question of so many years (or *hundreds* of years, if you like) for a man to work out *all* their properties. Then he *might* go to curves of the third degree. Say *that* took ten times as long (you see we have *unlimited* time to deal with). I can hardly imagine his *interest* in the subject holding out even for those; and, though there is no limit to the *degree* of the curves he might study, yet surely the time, needed to exhaust *all* the novelty and interest of the subject, would be absolutely *finite?* And so of all other branches of Science. And, when I transport myself, in thought, through some thousands or millions of years, and fancy myself possessed of as much Science as one created reason can carry, I ask myself 'What then? With nothing more to learn, can one rest content on *knowledge*, for the eternity yet to be lived through?' It has been a very wearying thought to me."—Lewis Carroll, *Sylvie and Bruno concluded*, p. 258.

ON THE NECESSITY OF PLACING THE SIGN × BEFORE THE MULTIPLICAND.

BY SARADAKANTA GANGULI.

AMONG many English writers of text-books on arithmetic and algebra there appears to exist a state of confusion regarding the use of the sign of multiplication. Here are three instances. Godfrey and Price's *Arithmetic* (1915) contains the following statement :

"The sign × means multiplied by ; thus 3 ft. × 7 = 21 ft., and (7 × 3) ft. = 21 ft. ; but 7 × 3 ft. is unmeaning" (p. 13). Yet, on page 147, we find the expressions

"$324 \times \frac{3}{10}$ miles" and "$20 \times \frac{240}{113}$ lbs.",

in which multipliers precede the corresponding multiplicands.

Again, in Workman's *School Arithmetic* (1903) we read : "The sign of multiplication is ×, which is read 'multiplied by' or 'times'". Accordingly, in this book, a concrete multiplicand is sometimes followed and sometimes preceded by a multiplier with the sign of multiplication between them.

In his *Arithmetic for Schools* (1894) Lock writes : "Thus 34 × 6 is an abbreviation for 34 multiplied by 6, which is read six *times* thirty-four".* (p. 16).

This confusion seems to be due to the previous existence of two different schools of thought regarding the use of the sign of multiplication. One school is represented by Wood,† Hind,‡ Peacock,§ A. De Morgan,|| and Chrystal,¶ who hold that the quantity which precedes the sign × is to be multiplied by that which comes after it. The other school is represented by Hamblin Smith $ and Thomas Lund,** who follow the Swiss mathematician Euler †† in maintaining that the multiplier should be written before the multiplicand and that the sign ×, when used, should, therefore, precede and not follow the multiplicand.

This state of confusion among writers on an exact science like mathematics is not desirable and calls for an early remedy which may be found in sticking rigidly to one of the above-mentioned two schools of thought. Although modern English writers evince a greater inclination towards the first school, the present writer thinks

* But Workman would read 34 × 6 as thirty-four *times* six.

† Wood's *Algebra*, revised by Lund (13th edition, 1848), p. 30, seems to contain Wood's view on the subject.

‡ Hind's *Algebra* (5th edition, 1841), p. 4.

§ Peacock's *Algebra*, vol. i. (1842), p. 5.

|| De Morgan's *Elements of Arithmetic* (1857), p. 26, Art. 53 ; but he also places the multiplier before the multiplicand in the statement "£2 = 2 × £1" (p. 139).

¶ Chrystal's *Algebra*, vol. i. (2nd edition, 1889), p. 11.

$ *A Treatise on Arithmetic* (4th edition, 1877), p. 17. But on page 128 the multiplier is written after the multiplicand in some cases.

** Wood's *Algebra*, remodelled by Lund (18th edition, 1878), p. 2.

†† *Elements of Algebra*, translated into English by Hewlett (4th edition, 1828), pp. 6 and 7.

that uniformity in procedure and the history of origin of the signs of algebraical operations demand strict adherence to the second school.

The modern practice of placing the sign of multiplication after the multiplicand owes perhaps its origin to the principle of "the attachment of the symbol of operation" or "operator" to the "subject" or "operand" on which it acts.*

As in the case of addition and subtraction the operand (*i.e.* the quantity on which the operation of addition or subtraction is to be performed) is followed by the sign of operation, so it may be argued with considerable force that also in the case of multiplication and division the operand (*i.e.* the multiplicand or the dividend, as the case may be) should, for the sake of consistency, be followed by the sign of multiplication or division. In that case the square root of a should be denoted by $a\surd$ and not, as at present, by $\surd a$. For here a is the operand, and the sign of operation of the extraction of the square root should follow and not precede it. The same principle of the attachment of the symbol of operation to the operand is followed also in Calculus and Quaternions where the symbol of operation precedes and does not follow the operand; *e.g.*

$$Dy, \quad \int f(x)dx, \quad qa.$$

It will thus be seen that if the practice of placing the sign × after the multiplicand is sought to be supported by the principle of the attachment of the symbol of operation to the operand, there is inconsistency in the application of the principle. Nor can the signs +, − and ÷ be placed, without disturbing the time-honoured convention regarding their position, before their respective operands as is the practice in Calculus and Quaternions. Hence it is advisable not to apply the above principle in connection with the signs of algebraical operations. In order to place the signs properly we should bear in mind the history of their origin and try to stick to their original use. Modern developments should not be allowed to interfere with their original use. They may, however, necessitate a new interpretation. When we take $\frac{1}{2}$ of a quantity we still *multiply* the quantity by $\frac{1}{2}$. Here the word *multiply* does not convey its original meaning; for it is meaningless to take a quantity half a time. Yet no one has seriously thought of changing the use of the word *multiply*.

The sign + or − is placed before a quantity which is to be added or subtracted. Although multiplication is now considered to be an operation distinct from addition, self-contained and self-controlled, it cannot be denied that it has grown out of addition and that originally it was regarded as a shortened process for repeated addition of the same quantity. To multiply a by 4 is the same thing as finding the sum $+a+a+a+a$, to use Chrystal's expression.† Here the quantity affected is a, as also indicated by the sign + before

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* Chrystal's *Algebra*, vol. i. (1889), p. 4.

† Chrystal's *Algebra*, vol. i. p. 11.

it. The figure 4 (called the multiplier) is not even present in the expression. Then why should the absent figure be affected with a new sign? The quantity which should have a sign before it is a and not 4. When the above expression is simplified by writing $+a$ only once, it becomes necessary that the number of times $+a$ is to be repeated should be indicated by 4. But that is no reason why, for the sake of simplification, the quantity which is affected with a sign should have the sign suddenly dropped and another quantity which did not originally occur in the expression should have a new sign prefixed to it. This new sign is the St. Andrew's cross $\times$, now used as a symbol of multiplication. How this use of the sign has sprung up we do not definitely know. Cajori holds * that Oughtred was the first to use $\times$ as the sign of multiplication. Suggestions for such a use of the sign could not come from the various ways † in which it had been previously used. Yet we are not quite in the dark regarding the matter. Considerable light is thrown on the subject by the following facts: ‡

(1) The present arrangement of the dividend and divisor in the process of division is due to the German mathematician Michael Stifel, who suggested it in 1545.

(2) To signify multiplication and division Michael Stifel suggested the use of the German capitals corresponding to M and D by the side of the symbols $+$ and $-$.

(3) The present sign $\div$ for division had been in use as an alternative sign for subtraction before it was adopted by the Swiss, Johann Heinrich Rahn, as a sign of division in 1659.

It will thus appear that the use of a less popular sign of subtraction for that of division (which is an abbreviated process of repeated subtraction of the same quantity) was very probably inspired by Stifel's suggestion for a symbol of division. This inference, together with the first of the three facts stated above, leads us to hold that Oughtred in his turn owed his suggestion for the sign of multiplication to the same source, and did away with the necessity of writing M by the side of the sign $+$ by turning the sign through half a right angle so as to avoid confusion with the sign of addition. The proper place for the sign of multiplication is, therefore, *before* the quantity which is to be taken a number of times indicated by the multiplier, *i.e. before* the multiplicand. Since the sign of multiplication is to connect the multiplicand with the multiplier, the

* *A History of Mathematical Notations*, vol. i. (1928), p. 265, Art. 231.

† *Ibid.* p. 252. These ways have been enumerated by Cajori as follows: "(1) in solutions of problems by the process of two false positions, (2) in solving problems in compound proportion involving integers, (3) in solving problems in simple proportion involving fractions, (4) in the addition and subtraction of fractions, (5) in the division of fractions, (6) in checking results of computation by the processes of casting out the 9's, 7's, or 11's, (7) as part of a group of lines drawn as guides in the multiplication of one integer by another, (8) in reducing radicals of different orders to radicals of the same order, (9) in computing on lines, to mark the line indicating 'thousands', (10) to take the place of the multiplication table above 5 times 5, and (11) in dealing with amicable numbers".

‡ Cajori, *A History of Mathematical Notations*, vol. i. p. 270.

multiplier should occupy the vacant place preceding the sign. Hence the order should be: (from left to right) multiplier, sign of multiplication, multiplicand. Although De Morgan advocates the placing of the sign × after the multiplicand, he, too, appears to support this very arrangement when he writes " *pa* signifies that *a* is taken *p* times ".*

The practice of placing the sign of multiplication before the multiplicand would be quite in keeping with the prevalent practice of placing the sign of division before the divisor. $35 \div 6$ means 6 is to be subtracted from 35 as often as possible. If the subtraction were to be performed only once, $35-6$ would express the fact. This explains why Stifel suggested to put the letter D by the side of − to signify division. Rahn improved upon the suggestion by using, as already stated, a less popular sign (viz. ÷) of subtraction. The adoption of this sign led necessarily to the abandonment of the use of the letter D. Thus the sign of division is placed before the divisor or the quantity repeatedly subtracted.

Following the practice of prefixing the proper sign before the quantity added or subtracted (whether once or repeatedly), we should place the sign $\surd$ before the quantity whose root is required. Owing to the absence of a second quantity in this case, the question of placing the sign before another quantity does not arise.

In a^m m is not a sign of operation; it cannot by itself imply any operation. Hence the propriety or otherwise of writing m to the right need not be considered.

The placing of the sign × before the multiplicand may create some difficulty in the case of Cantor's transfinite ordinal numbers. But this should not be allowed to stand in our way. For, in the case of transfinite ordinals, we have to revise some of our ordinary ideas. Where we have to admit that $\alpha+\beta$ is not equal to $\beta+\alpha$, we may also agree to reverse the ordinary order in the arrangement of the multiplier, sign of multiplication, and the multiplicand.

It is very desirable that an early decision should be arrived at regarding the proper position of the sign of multiplication.

SARADAKANTA GANGULI.

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883. " An amusing experience of formality occurred to him in connection with this novel [*Two on a Tower*]. It was necessary that he should examine an observatory, the story moving in an astronomical medium, and he applied to the Astronomer Royal for permission to see Greenwich. He was requested to state before it could be granted if his application was made for astronomical and scientific reasons or not. He therefore drew up a scientific letter, the gist of which was that he wished to ascertain if it would be possible for him to adapt an old tower, built in a plantation in the West of England for other objects, to the requirements of a telescopic study of the stars by a young man very ardent in that pursuit (this being the imagined situation in the proposed novel). An order to view Greenwich Observatory was promptly sent."—*The Early Life of Thomas Hardy*, ch. xi., p. 195. [Per Mr. P. J. Harris.]

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* De Morgan, *Elements of Arithmetic*, p. 19.

THEOREMS CONNECTED WITH FOCAL CHORDS OF A CONIC.

By E. P. Lewis.

1. *PSQ* is a focal chord of an ellipse and the normals at P and Q intersect at U.

Theorem I. *The locus of the foot of the perpendicular from U to PSQ is a similar coaxal conic.*

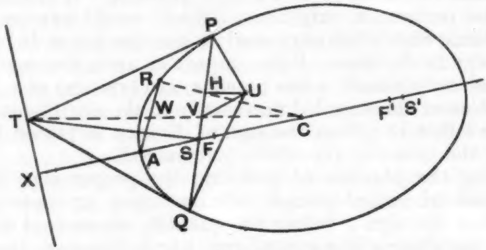

Fig. 1.

Let the tangents at P and Q meet at T: then T lies on the directrix and TS is perpendicular to PSQ. Draw UH perpendicular to PSQ and AR parallel to PSQ to meet the given conic at R and CVT at W. Since TU is the diameter of the circle $TQUP$, and UH, TS are perpendicular to PQ, V, the mid-point of PQ is also the mid-point of SH. But W is the mid-point of AR and hence C, H, R are collinear. Hence $CH/CR = CS/CA = e$, the eccentricity.

Hence the locus of H is a similar coaxal ellipse whose major axis is SS' and whose foci F, F' are such that

$$CF/CS = e = CS/CA\ ;$$

thus the directrices of this ellipse are the tangents at the vertices of the original ellipse.

Theorem II. *A hyperbola can be described with one focus at S' to touch the normals UP, UQ at P and Q.*

For $$S'Q - S'P = SP - SQ = HQ - HP$$

so $$S'P - HP = S'Q - HQ.$$

Hence P and Q lie on a hyperbola, foci H and S'; also since PU and QU bisect the angles HPS', HQS' respectively, PU and QU are the tangents at P and Q to the curve.

Theorem III. *As the chord PSQ varies, the locus of the second focus, H, of these hyperbolas is the conic locus of Theorem* I, *foci F and F'.*

Theorem IV. *Each conic of this family of hyperbolas passes through two fixed points on the axis.*

Let PN, QM, HK be the ordinates of P, Q, H. Then

$$\begin{aligned} S'Q - HQ &= AA' - SQ - SP \\ &= e(XX' - XM - XN) \\ &= e(XX' - XS - XK) \\ &= e\,.\,S'K. \end{aligned}$$

Further, $\quad S'F + S'F' = SS' = HF + HF' \quad$ (Th. I)

whence $\quad S'F - HF = HF' - S'F'.$

Also, $\quad S'F - HF = e(AS' - AK) \quad$ (Th. I)

$$\begin{aligned} &= e\,.\,S'K \\ &= S'Q - HQ. \end{aligned}$$

Hence F and F' lie on the hyperbola, foci H and S', touching UP, UQ at P and Q.

THEOREM V. *The latus rectum of each member of the family is equal to the latus rectum of the original conic.*

Reciprocate the system with respect to the point S'. The hyperbolas reciprocate into circles touching the two parallel lines f, f' which are the reciprocals of F, F'; thus the circles are all equal. Now

$$\frac{\kappa^2}{S'F'} - \frac{\kappa^2}{S'F} = \frac{\kappa^2}{ae(1-e)} - \frac{\kappa^2}{ae(1+e)} = \frac{2\kappa^2}{a(1-e^2)},$$

and

$$\frac{\kappa^2}{S'A'} + \frac{\kappa^2}{S'A} = \frac{\kappa^2}{a(1-e)} + \frac{\kappa^2}{a(1+e)} = \frac{2\kappa^2}{a(1-e^2)}.$$

Hence the original ellipse reciprocates into a circle whose diameter is equal to the diameters of the other circles. Thus in the original figure all these conics have equal latera recta.

THEOREM VI. If *P', Q' are the two remaining intersections of a hyperbola with the ellipse, then $P'Q'$ passes through S'.*

In the reciprocal figure the direct common tangents of the two equal circles are parallel, so in the original figure P', S', Q' are collinear.

2. THEOREM VII. *If the original conic is a parabola, the corresponding family is a family of equal parabolas with axes parallel to the axis of that parabola. Each curve of the family passes through a fixed point F on the axis, which is such that $AF = 2AS$, where A is the vertex of the original parabola and S is its focus. The locus of the foci of these parabolas is an equal parabola whose axis coincides with the axis of the given parabola and whose focus is F.*

As before, draw AR parallel to PSQ and HK perpendicular to the axis. Let F be the point on the axis such that $AF = 2AS$. The tangents TP, TQ are in this case at right angles, so TU bisects PQ at V and is accordingly parallel to the axis. Now W is the mid-point of AR, and V is the mid-point of SH, hence RH is parallel to AS and consequently equal to it. Thus, RH is equal to SF and $SFHR$ is a parallelogram. So, if RN is the ordinate of R,

$$FH = SR = XN = AK.$$

Whence the locus of H is an equal parabola, focus F, whose directrix is the tangent at the vertex A of the original parabola.

Again, if UY is perpendicular to the axis and PD is perpendicular to UY,
$$\angle PUD = \angle PTU = \angle PQU = \angle PUH;$$
thus the triangles PHU, PDU are congruent. Hence UP and UQ are the tangents at P and Q to a parabola, focus H, directrix UY.

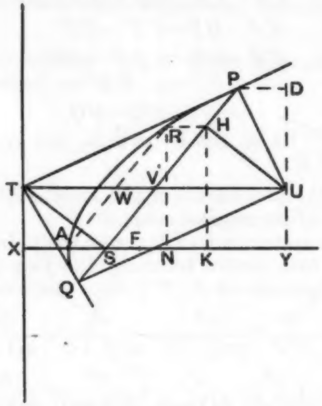

Fig. 2.

The semi-latus rectum of this parabola is the perpendicular from H to UY, which, by symmetry, is equal to the perpendicular from S to TX, which is $2AS$ and so is equal to AF. Hence
$$FY = AK = HF.$$
Thus F also lies on this parabola. Every member, therefore, of the family has its latus rectum equal to the latus rectum of the original parabola, has its axis parallel to the axis of that parabola, and passes through the fixed point F.

3. The Cartesian equation of the family of hyperbolas in the preceding work may be of interest. If the ellipse is
$$x^2/a^2 + y^2/b^2 = 1,$$
of eccentricity e, and α, β are the eccentric angles of P and Q, and for brevity, $t = \tan \frac{1}{2}(\alpha + \beta)$, then the equation of the family is
$$b^2x^2 + 2abtxy + (b^2t^2 - a^2e^2)\,y^2 + 2a^2bte^3y - a^2b^2e^4 = 0.$$
The reduced equation referred to the axes of the conics is
$$x^2 \Big/ \left(\frac{a^2e^4}{(1+t^2)^2}\right) - y^2 \Big/ \left(\frac{b^2e^2}{1+t^2}\right) = 1.$$

E. P. L.

884. The student must not believe that theorems have been invented or perfected by the methods in which it is afterwards most convenient to deduce them. The march of the discoverer is generally anything but on the line on which it is afterwards convenient to cut the road.—De Morgan, *Differential and Integral Calculus*, p. 171. [Per Dr. G. J. Lidstone.]

MATHEMATICS IN THE SCHOOLS.

TWO REPLIES TO MR. A. N. HICKLING.

I. By E. H. Lockwood.

Mr. Hickling's article on "Mathematics in the Schools" is a challenge to all those of us who are following the modern tendencies in the teaching of Mathematics. Many good and sufficient reasons have been advanced why Trigonometry, Mechanics and even Calculus should be made available to the average boy before he has passed School Certificate. Experiments have been made, suitable text-books have been published, and the idea has been recognised by examining boards by the introduction of "Additional Mathematics" as an optional subject. Mr. Hickling asks us to go back on all this, to damp down the "unfortunate demand for Higher Mathematics", and to restrict the work to the Arithmetic, Algebra and Geometry of thirty years ago. The burden of proof therefore lies with him, and we are entitled to ask whether he is facing the facts.

In the first place he ignores the fact that most boys' mathematical education ends at the age of 16 or 17, so that they will never be able to embark on the more advanced course, and will leave school with no conception of the scope of Mathematics or of its relation to the modern world.

He furthermore implies that the elementary subjects can be completely mastered by that age, and asserts that a thorough and complete knowledge of them is a necessary preliminary to the study of even the elements of Trigonometry, Calculus, Statics and Dynamics. Numbers of teachers will testify that this is not true. Trigonometry can be easily and profitably taught to boys of 14 and 15, and Statics may be begun at almost any age. Dynamics and Calculus are more difficult, probably because so few boys have been brought up to understand the idea of functionality, but there is at least no reason to regard them as impossible.

Does Mr. Hickling realise that the system he advocates has been tried and found wanting? If he will enquire from a group, selected at random, of people who were brought up that way, he will find that few of them gained either interest or confidence, still less that appreciation of logical structure which he rightly regards as an important part of the value of the subject. Even for the favoured few whose mathematical education is continued beyond the School Certificate age, the restriction he advocates would mean a loss of interest and a reduction of the amount of ground eventually covered.

Surely our real problem is to find what is the desirable minimum of work in the elementary subjects, considered both for their own sake and as leading on to the so-called "Higher Mathematics".

E. H. L.

Felsted School, Essex.

II. By Hilda W. Oldham.

It is true, as Mr. Hickling says in the July *Gazette*, "It is not until an advanced state of study and thought is reached that fundamental principles and assumptions come up for revision, that axioms and hypotheses can be placed upon a sound basis of mathematical argument". So out of his own mouth Mr. Hickling provides a most excellent reason why the elements of mathematics should be dealt with from an intuitive rather than from a logical point of view. The boy or girl begins Geometry at about eleven years of age; human intelligence does not reach its full development until the age of approximately sixteen; at eleven years old the child is not able to detect the fine points of logic that delight the mathematical expert. It may be a danger, then, for the brilliant young mathematician to be put in charge of the young beginner in Geometry, for he may be placing over his pupil's nose logical spectacles that obscure the intuitive vision he has.

Mr. Hickling tells us "the whole essence of mathematics is to take nothing on trust, but to search and search again for any suspicion of error". Does he suggest that in the lower school geometry everything be pushed to its logical beginning, and how does he propose to do it? In Euclid's text-book there is a so-called logical sequence, but there is no mathematician who would pretend that it is not full of assumptions.

I should have thought that in good school mathematics a very great deal is taken on trust, and rightly so.

It is possible—and here I speak from experience—for very useful work to be done by "dilettante" mathematicians, who have more knowledge of child psychology and less of the higher branches of mathematics than the brilliant young Cambridge mathematician.

I must confess that I find it very difficult to appreciate the division that Mr. Hickling emphasises into utilitarian and educational values of mathematics. Perhaps more vague talk is indulged in on the value of education for its own sake than about anything else.

Is not the purpose of all education to endow life with a fuller meaning? There is a danger that the pedagogue may over-emphasise technique and mistake it for education. Mr. Hickling suggests that up to matriculation Arithmetic, Algebra and Geometry are the only mathematics that should be taught to the average child. His point of view is, I think, influenced by this over-emphasis of technique at the expense of education. It is quite possible to teach very simple heights and distances, involving only the use of the tangent before the harder theorem of Pythagoras has been reached. Children are delighted to find the height of buildings and monuments by this simple trigonometry, and approach the other trigonometrical ratios with a much less chance of confusion, having mastered one to begin with. The simple parts of the Calculus can be taught to a School Certificate Form, and an interest in the Calculus may thus be aroused.

Probably twenty or thirty years ago teachers of the Calculus did fail to give the reasons for equating to zero the first derived function, and to distinguish clearly between maximum and minimum values, but surely most people to-day approach functions and derived functions by graphs, and leave no doubt in the pupil's mind as to whether the function has reached its maximum or minimum value.

I have met many people who have expressed resentment at the narrowness of their mathematical education. Here are some specific instances.

(i) A woman forester of good mental ability had been educated at a high school where Trigonometry was reserved for the mathematical specialists in the VIth Form. A knowledge of the tangent, sine and cosine, that could easily have been given her in her matriculation year and the one preceding it, was all she needed. She had to waste time and effort from her strenuous work in reading up and getting skill in manipulation of this work that I feel should have been part of her school education. The mathematician's answer to this may be, " Such easy work can very readily be crammed up after leaving school ", but to the person with little mathematical ability or interest such work is difficult and burdensome, and takes time from the full enjoyment of life.

(ii) A man who is a lecturer in a well-known scientific institution in England is held up in his research because of his lack of mathematics to do the necessary statistics his work needs.

As a teacher of elementary statistics, I know very well that no school mathematics alone can enable a man to follow the intricacies of statistical methods. But a knowledge of elementary Calculus, such as can be given in the School Certificate year, is a very great aid in working out some of the simpler problems in statistics.

(iii) A working engineer sometimes comes to me about simple problems in mechanics. He gets his results from tables, but he would find greater zest in his work if he could understand the general principles which govern the formulae he is using.

Of course, no one would attempt to teach the elements of the Calculus until the easy parts of Algebra and Geometry had been mastered. But much time has been wasted in the past on such things as long division, long H.C.F's., tedious identities. Good Certificate Forms can readily be taught the elements of probability, without which it is dangerous to set out on some scientific research. It is surprising the credulity existing amongst partly educated people regarding such things as astrology and occultism generally, a credulity which would be considerably shaken if the happenings could be compared with chance happenings arrived at by comparatively simple mathematical calculation.

There would be more to be said for this thoroughgoing teaching of mere mathematical technique if fifty per cent. of pupils could take it in; but such mathematics is so remote from the life and interests of the average boy or girl that it passes in at one ear and out at the other without having made any appreciable impression. Everyone must appreciate Mr. Hickling's fear lest we overburden

the mental capacity of the young, but when curiosity and real interest are aroused mental fatigue will be considerably lessened. Any contacts school mathematics can make with life must arouse such interest, for if Jones is not going to be an engineer his friend Brown may be, and Calculus will be of value to him. Moreover, it adds to the zest of life to find the height of a building, and to argue with your friend on its probable height before doing so. Can anyone doubt that the teaching of Mathematics on this suggested broader basis must create not only greater interest amongst the many, but that it will be productive of more alert students who, having gazed at the interesting goods in the window of the mathematics shop, will want to examine them more closely and to judge of the validity of their supposed virtues. H. W. O

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885. "But you speak of instruction, and of a profession; are you an adjunct to the provincial corps as a master of the noble science of defence and offence? Or perhaps you are one who draws lines and angles under the pretence of expounding the mathematics."—Cooper, *The Last of the Mohicans*, ch. ii, p. 16.

886. Your trousers will have lost all trace of those impeccable creases which are the two parallels of snobbish geometry.—M. Dekobra, *The Madonna of the Sleeping Cars.* [Per Mr. H. Berry.]

887. *A Companion for S. Pedito.*

Mr. Faraday, St. Henry Exhibition,
c/o Messrs. . . . , Royal Albert Hall.

True copy of label on parcel of gramophone records. [Per Mr. F. Beames. *The Morning Post*, Sept. 23, 1931, transcribed the label under the heading "Our Alert Business Men"; for S. Pedito see St. John Lucas, *Saints, Sinners, and the Usual People.*]

888. People add other qualities to beauty—sublimity, human interest, tenderness, love—because beauty does not long content them. Beauty is perfect, and perfection (such is human nature) holds our attention but for a little while. The mathematician who after seeing *Phèdre* asked: '*Qu'est-ce que ça prouve?*' was not such a fool as he has been generally made out.—W. Somerset Maugham, *Cakes and Ale*, p. 123. [Per Mr. J. Buchanan.]

889. A little while ago I read in the *Evening Standard* an article by Mr. Evelyn Waugh in the course of which he remarked that to write novels in the first person was a contemptible practice. I wish he had explained why, but he merely threw out the statement with just the same take-it-or-leave-it casualness as Euclid used when he made his celebrated observation about parallel straight lines.—W. Somerset Maugham, *Cakes and Ale*, p. 188. [Per Mr. J. Buchanan.]

890. Just as an integer is a complete figure with no vulgar fraction, decimal, or anything else about it that prevents you from taking its measure simply and directly, a man of integrity is one whose moral make-up is sound and complete. In particular, there is nothing fractional or incalculable about his honesty; he always gives you value for money.—From an advertisement, May 1931. (An advertisement for Standard cars, *The Sketch*, May 13, 1931, p. xxiii.) [Per Mr. J. Buchanan.]

THE TRAINING OF MATHEMATICS TEACHERS.

[When the activities of the International Commission on the Teaching of Mathematics were suspended by the War, an enquiry into the training of school teachers was in progress, and when the Central Committee of the Commission decided that whole-hearted cooperation was again possible, advantage was naturally taken of the fact that there was interrupted work to be resumed. Some of the participating Sub-Commissions had already presented reports; to the others a revised version of the questionnaire on which reports were to be based was issued. The reports were considered by Prof. Gino Loria, the Italian mathematician and historian, whom we are proud to number among the honorary members of our own Association, and he presented at the Zürich Congress a comprehensive survey of the whole subject.

Communication with the United Kingdom was made through Prof. E. H. Neville—who has since been offered a seat on the Central Committee—and the following Report was compiled for the Commission by the Board of Education, for whose cooperation, as willing as it was indispensable, the Commission is most thankful.]

I. General Preparation of Candidates.

Teachers of Mathematics will normally have spent their early years in a Public Elementary School, the Preparatory Department of a Secondary School, or a private Preparatory School. From either of the first two transference to a Secondary School takes place at about 11 +, but the private Preparatory School generally sends pupils on to the Public Schools (Secondary) at about 13 +. Transfer from the school to the University generally takes place at eighteen to nineteen years of age.

At the Secondary School a general course is followed up to the age of about sixteen, when the First Examination is taken. The curriculum includes the study of English, Scripture, Geography, History; one, two or three foreign languages; Mathematics always, and Science generally. Of the foreign languages French is the first choice in most cases, Latin the second, and Greek or German the third. The number of pupils taking German is now tending to increase.

From the passing of the First Examination until entrance to a University a more specialized course is followed. Those who will ultimately be taking a degree in Mathematics will commonly have spent most of their time during these years in reading as their principal subjects Mathematics and Physics, or Mathematics, Physics and Chemistry, or less frequently Pure Mathematics; as subsidiary subjects they generally devote some time to English and to a modern foreign language. In the older universities of Oxford and Cambridge some Latin is required from candidates for a mathematical degree, but the Latin done in preparation for the First Examination at sixteen years of age meets the requirements. In the more recently established universities Latin is not required from those who are taking a mathematical degree.

Of the Secondary teachers in England who are trained, the great majority have spent four years in one of the universities and have taken the University Diploma in Education. The first three years

are occupied in taking the University degree, which may be an ordinary or pass degree in a group of subjects or an Honours degree in a much more limited number of subjects, and often only one subject. The fourth year is given up wholly to the professional training, which is entirely divorced from the academic or scientific training, and may, indeed, be taken at a different university.

It should be made clear that while opportunities for training exist, there are no regulations laying down that training, or indeed a University degree, is necessary for teachers of Secondary schools. The following figures may, however, be interesting. In 1913, of the men teaching in Secondary schools, 71·6 per cent. were graduates, 37·5 per cent. were trained and 27·9 per cent. were trained graduates; the corresponding figures for the women being 52·3 per cent., 47·4 per cent. and 29·7 per cent. In 1931, of the men, 83·6 per cent. were graduates, 49 per cent. were trained and 44 per cent. were trained and graduates; corresponding figures for the women being 65·5 per cent., 46 per cent. and 39 per cent. It will be seen that nowadays the great majority of the teachers are graduates. The Honours graduates in Mathematics will usually have a good knowledge of Applied Mathematics, Mechanics, and Physics, but not necessarily of Philosophy, History or foreign languages. Of the pass graduates the majority—those who hold a Science degree—will have some knowledge of the same subjects, but the others, with an Arts degree, may have combined Pure Mathematics with literary subjects.

Four Year students, *i.e.* those who are taking a three-year degree course of academic study, followed by a year of professional training in a University Training Department, are eligible for grants towards their tuition and maintenance from the Board of Education provided that they intend on the completion of their course to teach in state-aided schools. The tuition grant for the degree course varies with the fees charged and is, as a rule, sufficient to meet the whole of the fees. The tuition grant for the postgraduate training is £35; at some University Training Departments this covers the charges, at others the students pay a fee. The maintenance grant in each of the four years is £43 for a man and £34 for a woman, if resident in a college or recognised hostel; £26 for a man and £20 for a woman, if not so resident.

II. Theoretical Scientific Teaching, *i.e.* The Degree Course.

The preparation for the degree in the case of Mathematics (pure, applied, and mechanics) is carried out almost entirely by means of lectures, exercises, and preparation. The number of lectures in these subjects would vary considerably, but in many cases would be about ten to twelve per week. The time given to exercises and preparation would depend entirely on the student. Usually in the degree course little or no attention is given to the Foundations of Mathematics, the History of Mathematics, or practical work in Mathematics.

Students taking the ordinary or pass degree commonly take

Physics, and sometimes Physics and Chemistry, as well as Mathematics, though certain other subjects could be chosen in place of them. In these branches of Science the teaching is by means of lectures, exercises, preparation, combined with a considerable amount of experimental work in the laboratories.

This theoretical preparation is tested by the University degree examinations. There is no Government examination.

III. Professional Training.

The professional training may vary in details in the different University departments, but normally students will have had courses in :

The Principles of Education.
General Methods of Teaching.
Educational Psychology.
Educational Hygiene.
The History of Education.

In addition the course must include practice in teaching under supervision in a school, and students who have not had previous teaching experience must spend at least twelve weeks in such practice.

The actual amount of instruction in the methods of mathematical teaching given by the University Training Department varies very considerably. The lessons may be given by a member of the university staff, or by someone who is teaching or has taught in schools and is specially engaged for this purpose. The extent of such courses may vary from very little to a thorough discussion of methods, applicable to pupils to the age of sixteen.

The period of practical training may be made up of three separate months at different points of the year of training, and these three months may be spent at different schools. On the other hand, the period of training may be one of three months taken consecutively in one school. During the period of practical training the student commonly is treated in the same way as a junior member of the teaching staff, with similar privileges, etc. The student who intends to teach Mathematics will be under the immediate direction of the Head Master (if he is a mathematician) or of the Chief Mathematical Master, and will spend his time in hearing lessons, giving lessons in the presence of the regular teacher, and, later on, taking complete charge of classes. In a large school where there may be four or five Mathematical teachers there will be plenty of opportunity for him to see how different parts of the subject are presented, and, in the case of Applied Mathematics, of seeing to what extent practical work is encouraged.

There is no study of Educational Legislation except in so far as it may be dealt with under the History of Education.

The professional training may be tested by a definite written examination or by essays. The diploma may be awarded partly on the mark given for these examinations or essays and partly on the

mark given on the student's ability to teach as shown during the three months' practice. On the other hand there may be no written examination, in which case the award may be made in part on the student's written work throughout the year's training or on a thesis presented by the student.

As to the current opinion on the value of these courses, there is no doubt that a quarter of a century ago they were not regarded with particular favour. The figures previously quoted as to the proportion of trained teachers now in the schools gives some idea of the changed position in this respect.

The methods of training referred to hitherto has been that of the University Training Departments. There are, however, two or three other methods that should be mentioned.

(i) The Training Colleges, of which there are a great many, aim primarily at the training of teachers for the Elementary Schools, but a number of these obtain posts in Secondary schools either at the close of training or, later on, by transfer. The general lines of the training are similar to those described, except that there may be no special attention given to Mathematics.

(ii) The Board of Education may recognize arrangements for the training of persons who have University Degrees in courses of not less than a year in Secondary schools. Any such arrangements must provide for a systematic study of the practice and principles of teaching, and the school must satisfy the Board that it can provide a course suited to the needs and capacity of the particular student concerned. Few schools carry on this form of training, and in no school would there be more than one or two students.

IV. Subsequent Improvement.

Courses in Mathematics for Secondary school teachers are held in the summer vacation. There is no compulsion to attend, but there is considerable demand. In 1931, for example, something like 200 teachers applied, and of these 80 were selected for the courses.

It is not usual to grant a term or terms leave, even after some years teaching, in order to keep in touch with developments in the subject; there is, however, a Mathematical Association with numerous branches all over the country, and with a periodical (see § VI below), which help in this direction.

Arrangements may be made by which a teacher may pay an observation visit to other schools where the subject is unusually well organised or taught.

It is quite common for teachers to produce text-books on the subject, for teaching purposes. It is unusual for them to occupy themselves in research or to contribute to any great extent to the sum total of knowledge of the subject. Nor is it common for teachers in Secondary schools to advance to University posts. One or two well-known mathematicians, however, have spent some part of their career in school teaching, as, for example, F. S. Macaulay and W. P. Milne.

V. Duties and Responsibilities.

A teacher whose main subject is Mathematics may spend part of his time in teaching any other subjects. If he is not fully occupied with his own subject he most commonly helps with the Physics.

Mechanics of an elementary kind may be taught either by teachers of Science as an experimental subject in the early part of the Science course or, from the age of sixteen to eighteen, as a branch of Mathematics by the Mathematical teachers.

A preliminary course in descriptive geometry usually precedes the deductive course. It is taken by the same teacher and does not, as a rule, amount to a great deal.

The chief Mathematical teacher in any school is responsible for formulation of programmes of study and for co-operating with the heads of the Science and other departments. Such programmes, etc. will be always subject to the approval of the Head Master or Mistress.

Teachers of Secondary schools are not required by law to have any qualifications, though they are in most cases graduates and are recruited by the methods already indicated.

The great majority of teachers are paid salaries in accordance with the Burnham Scale, and are eligible for the Government pension.

VI. Bibliography.

Of the very few books dealing with the teaching of Mathematics, or of particular branches of Mathematics, at the stage with which this report is concerned, the best known are :

B. Branford. *A Study of Mathematical Education*, xii, 432 ; 1921.

C. V. Durell. *The Teaching of Elementary Algebra*, viii, 136 ; 1931.

C. Godfrey and A. W. Siddons. *The Teaching of Elementary Mathematics*, xii, 332 ; 1931.

T. P. Nunn. *The Teaching of Algebra (including Trigonometry)*, xvi, 616 ; 1914. (A running commentary on two volumes of *Exercises in Algebra (including Trigonometry)* compiled by the same author.)

F. W. Westaway. *Craftsmanship in the Teaching of Elementary Mathematics*, xvi, 666 ; 1931.

At meetings of the British Association for the Advancement of Science, discussions were opened by J. Perry on the Teaching of Mathematics (1901) and the Teaching of Elementary Mechanics (1905) ; the verbatim reports of these discussions are still of interest.

The Mathematical Association, a society devoted to the improvement of the teaching of Mathematics in schools, issues from time to time special reports drawn up by committees which often include

inspectors and university teachers in addition to school teachers. The longest of these reports hitherto published are :

The Teaching of Geometry in Schools, iv, 74 ; 1923 (3rd ed. 1929)

The Teaching of Mechanics in Schools, 84 ; 1930.

The Teaching of Arithmetic in Schools, 82 ; 1932.

The Mathematical Association publishes also *The Mathematical Gazette* (Editor : T. A. A. Broadbent, 2 Buxton Avenue, Reading), which includes articles of interest to school teachers, and reviews not only of school and university text-books but also of the most advanced mathematical treatises, foreign as well as native ; five numbers compose an annual volume of about 400 pages. This is the only periodical designed for the mathematical teacher, but relevant articles and reviews appear occasionally in two weeklies, *Nature* and *The Times Educational Supplement*. The contact with Physics brings Mechanics on the experimental side into *The School Science Review*, the journal of the Science Masters' Association.

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891. In 1727 he witnessed the burial of Newton, and was amazed by the magnificent honours paid to scientific genius. The body, exposed by torchlight on a State bed, was borne to Westminster Abbey, followed by a long procession, including the Lord Chancellor and Ministers of the Crown. . . .

In later years he withdrew some of his enthusiasm : " I thought in my young days that Newton's fortune was made by his outstanding merit. I had imagined that the Court and City of London had acclaimed him as Master of the Mint of the Realm. Not at all. Isaac Newton had an agreeable niece, named Mrs. Conduit. She took the fancy of Halifax, the Lord High Treasurer. The infinitesimal calculus and gravitation would have availed him nought had it not been for a pretty niece ".—A. Maurois, *Voltaire*, p. 39. [Per Mr. E. H. Lockwood.]

892. O the grey dull day ! It seemed a limbo of painless patient consciousness through which souls of mathematicians might wander, projecting long slender fabrics from plane to plane of ever rarer and paler twilight, radiating swift eddies to the last verges of a universe ever vaster, farther and more impalpable.—J. Joyce, *Portrait of the Artist*, 1916, p. 223.

893. The moon, perhaps, has shrunken a little. . . . The crescent at evening still startles the soul with its delicate flashing. But the mind works automatically and says : " Ah, she is in her first quarter. She is all there, in spite of the fact that we see only this slim blade. The earth's shadow is over her ".—D. H. Lawrence, *Assorted Articles ; Hymns in a Man's Life*, pp. 156-7. [Per Mr. Frank Robbins.]

894. " Well may I say to Hector, as Sir Isaac Newton did to his dog Diamond, when the animal flung down the taper among calculations which had occupied the philosopher for twenty years, and consumed the whole mass of materials—Diamond, Diamond, thou little knowest the mischief thou hast done ! "—Scott, *The Antiquary*, ch. xxii. [Per Mr. R. O. Street.]

895. In his wretched life of less than twenty-seven years Abel accomplished so much of the highest order that one of the leading mathematicians of the nineteenth century (Hermite, 1822-1901) could say without exaggeration, " Abel has left mathematicians enough to keep them busy for five hundred years ". Asked how he had done all this in the six or seven years of his working life, Abel replied " By studying the masters, not the pupils ".—E. T. Bell, *The Queen of the Sciences*, (1931) p. 10.

MATHEMATICAL NOTES.

1049. *The Gauss-Lucas Theorem.*

One of the classic theorems of Algebra is the Gauss-Lucas theorem, which states that the smallest convex polygon containing in or on its boundary all the roots of an algebraic equation $f(z)=0$ will also contain in or on its boundary all the roots of the derived equation $f'(z)=0$. In this theorem it is understood that $f(z)=0$ has complex coefficients, and that the root $a+bi$ is placed at the point (a, b), just as complex points are ordinarily graphed.

The first proof of this theorem was given by Lucas, and was of a physical or mechanical, rather than strictly mathematical, nature. Three algebraic proofs have been published in the *Annals of Mathematics*, the first in ser. 1, vol. 7, 1892-3, by Bocher, the second by Hayashi, in ser. 2, vol. 15, 1913-4, and the third by Irwin in the following volume. Irwin's proof is practically the same as Bocher's. Hayashi's paper gives references to earlier work on the theorem.

The present note gives a simple proof based on a linear transformation of $f(z)=0$. It is, of course, well known that, in the z-plane, a translation and rotation of the axes can be effected by a transformation $w=az+b$. Let this transformation replace $f(z)=0$ by $F(w)=0$. It is also seen that the same transformation replaces $f'(z)$ by $F'(w)$, except for a constant factor, and it is this feature that makes the transformation permissible in any problem where we are concerned simply with the relative position of the roots of an algebraic equation and those of its derived equation. This fact has been mentioned before, for example, by Heawood, in the *Quarterly Journal of Mathematics*, vol. 38, 1906, page 84, but it has not been used extensively.

The Gauss-Lucas theorem may now be proved as follows. If it were false it would continue to be false after a translation and rotation. That is, the assumed root r of $f'(z)=0$ which lies outside the smallest convex polygon P enclosing all the roots of $f(z)=0$ may be taken as O, and P, from its convex property, may be taken entirely on one side of the y-axis. For the side of P which is closest to r can, by a rotation, be made parallel to the y-axis. Now if O is a root of $f'(z)=0$, the coefficient of z in $f(z)=0$ must vanish, and the relation $\Sigma 1/z_i=0$ must hold, the z_i being the roots of $f(z)=0$. Writing $z_i=x_i+y_i i$, rationalizing each denominator and equating the real part to zero, we have $\Sigma x_i/(x_i^2+y_i^2)=0$. This implies that not all the x_i can have the same sign, and our assumption that the Gauss-Lucas theorem is false is not tenable.

The principle employed in this note can, I believe, be used advantageously in the treatment of a number of other theorems concerned with relations between the roots of $f(z)=0$ and $f'(z)=0$, among them certain theorems of Walsh, Nagy, Laguerre and H. B. Mitchell. I hope to indicate this application in a later paper.

RAYMOND GARVER.

University of California, Los Angeles.

1050. *A certain expansion in the theory of determinants.*

This note arose out of a problem in the tensor calculus, and being of a determinant nature, it may be of some general interest.

Let g_{ij}, Ω_{ij} be the constituents of two symmetric determinants of order n, and let g denote the value of the determinant $|g_{ij}|$, and G^{ij} the minor of g_{ij} in $|g_{ij}|$. Defining $g^{ij}=G^{ij}/g$, and using the summation convention for a repeated index, *i.e.* writing

$$a_{i\alpha}b^{j\alpha}=\sum_{\alpha=1}^{n}a_{i\alpha}b^{j\alpha},$$

we have $$g^{i\alpha}g_{j\alpha}=\delta_j^i=0,\ 1 \text{ according as } i\neq j,\ i=j.$$

Define the constituents (g'_{ij}) of a new determinant by

$$g'_{ij}=g_{ij}+\lambda\Omega_{ij},$$

and define g', g'^{ij} for this determinant as above.

The problem is to express g'^{ij} as an infinite series in λ, this form being evident from $g'^{ij}=G'^{ij}/g'$, assuming $1/g'$ to be expanded in terms of λ.

Assume $$g'^{ij}=k_0^{ij}+\lambda k_1^{ij}+\lambda^2k_2^{ij}+\ldots+\lambda^rk_r^{ij}+\ldots .$$

Substituting in the identity $g'^{ik}g'_{jk}=\delta_j^i$, and equating coefficients of λ, we have

$$\begin{aligned}&k_0^{ik}g_{jk}=\delta_j^i,\\ &k_0^{ik}\Omega_{jk}+k_1^{ik}g_{jk}=0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,\\ &k_{r-1}^{ik}\Omega_{jk}+k_r^{ik}g_{jk}=0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,\end{aligned}$$

i.e. writing $\Omega_j^i=g^{i\alpha}\Omega_{\alpha j}$, we have

$$k_r^{ij}=-k_{r-1}^{i\alpha}\Omega_\alpha^j,\quad r\geqslant 1.$$

Also, the first equation gives $k_0^{ij}=g^{ij}$, and hence

$$k_r^{ij}=(-1)^r\Omega_{\alpha_1}^{\ i}\Omega_{\alpha_2}^{\ \alpha_1}\ldots\Omega_{\alpha_r}^{\ \alpha_{r-1}}g^{\alpha_rj},\quad k_0^{ij}=g^{ij},$$

where each repeated index indicates a summation.

This is a surprisingly simple result, for it includes no minors of $|\Omega_{ij}|$, and only the first minors of $|g_{ij}|$. In the application to such problems as the small deformation of quadratic forms, λ is usually small, so that λ^2 may be neglected, and we have

$$g'^{ij}=g^{ij}-\lambda g^{i\alpha}g^{j\beta}\Omega_{\alpha\beta}.$$

The above expansion is sometimes useful in evaluating symmetrical functions of the second minors of a determinant.

For example, to evaluate $\Omega_{\alpha\beta}G^{ij.\alpha\beta}$ where $G^{ij.\alpha\beta}$ is the second minor of $(g_{ij}, g_{\alpha\beta})$ in $|g_{ij}|$.

It is easily seen that, with the above notation,

$$G'^{ij}=G^{ij}+\lambda\Omega_{\alpha\beta}G^{ij.\alpha\beta}+\ldots .$$

But we also have $G'^{ij}=g'g'^{ij}$, and $g'=g(1+\lambda g^{\alpha\beta}\Omega_{\alpha\beta}+\ldots)$.

Hence, equating coefficients of λ,

$$\Omega_{\alpha\beta}G^{ij.\alpha\beta} = g\,(g^{ij}g^{\alpha\beta}\Omega_{\alpha\beta} - g^{i}\ \ g^{j\beta}\Omega_{\alpha\beta}).$$

We may remark that the above work could be repeated on similar lines when the given determinants are not necessarily symmetric. A. G. WALKER.

1051. *Square Roots and others.*

With reference to my article on this subject which appeared in the May number of the *Gazette* of this year, it would appear that the iterative process for finding square roots of numbers dates much further back than the time of Newton, for Sir Thomas Heath in his *Manual of Greek Mathematics* states on p. 423 that Heron of Alexandria (some 1600 years before Newton) described in his *Metrica* the getting of the first arithmetical mean of the arithmetical and harmonical means for this purpose and said that the process might be repeated. Curiously enough Heron's examples quoted by Sir Thomas are given by the aid of vulgar fractions but in a rather inconvenient form, and there is no indication of the actual *process* for carrying out the *principle* as I have given it, any more than there is in Whittaker and Robinson's *Calculus of Observations*. The very small step which I have made has consequences advantageous out of all proportion, so much so that I think I may fairly distinguish between the *principle* of iteration and the *process* with the best possible start, vulgar fractions and numerator differences as indicated as I have given it. The numerator difference is so quickly found that where the slide rule shows several vulgar fractions all apparently equally perfect, the finding of this difference in a few minutes shows which is the most accurate. Thus 37/6 is a fairly close value for $\sqrt{38}$, but 450/73, 530/86, 610/99 are better and are all perfect as near as the slide rule will show. If however they are tested by their numerator differences, the two elements obtained from 450/73 are found to be 202502 and 202500, differing by less than 1 in 100000, a degree of closeness not approached by the other two. With so close a start either first mean is only about 1 in 10^{11} in error and the second 1 in 10^{22} or less, and this second result requires no more extended work than one division sum with 405002 as divisor. C. V. BOYS.

1052. *Note on square roots.*

I wish, in this note, to point out a connection between the square root method presented by Dr. C. V. Boys in the *Mathematical Gazette*, XVI, May 1932, pages 111-115, and the theory of continued fractions. The value of establishing such a connection lies in the fact that it seems to give a better control of the accuracy of the approximations than is furnished by Dr. Boys, or by earlier writers on the same method.

It is well known that, if $\sqrt{N}$, where N is a positive integer, be expanded into a continued fraction, a cycle of partial quotients will

appear. If there are c partial quotients in the cycle, and if p_n/q_n represents, as usual, the n-th convergent of the continued fraction, it may be proved * that

$$\frac{p_{2mc}}{q_{2mc}}=\frac{p_{mc}^2+Nq_{mc}^2}{2p_{mc}q_{mc}},$$

where m is any positive integer.

Now if a/b is any rational approximation to $\sqrt{N}$, the iterative method gives as a second approximation $\frac{1}{2}(a/b+Nb/a)$, which reduces to $(a^2+Nb^2)/2ab$. Thus, if $a=p_{mc}$, $b=q_{mc}$, the iterative method simply proceeds from the mc-th convergent to the $2mc$-th convergent. And it is then known at once that we have a rational approximation which is in error by less than $1/q_{2mc}q_{2mc+1}$ and, *a fortiori*, by less than $1/(q_{2mc})^2$. Further, we have the best possible rational approximation unless we are willing to employ fractions with larger denominators.

In Dr. Boys' example, $\sqrt{10}$, which is equal to the continued fraction $3+\frac{1}{6+}\,\frac{1}{6+}\dots$, we have $c=1$. Thus if our first approximation is a convergent, the successive approximations will be convergents without any further worry on our part. The accuracy of each of them can be estimated at once. And there is surely no more natural way to obtain a good first approximation than by the use of a reasonable number of convergents in the continued fraction expansion of the radical. Thus the second convergent to 10 is $3\frac{1}{6}$. Of course, in any case where $c=1$ we may as well start with the first convergent; successive iterations will then give the second, fourth, eighth, convergents, and so on.

If the cycle is not as simple as this, it is necessary to be a little more careful in choosing the first approximation if we wish to make all the approximations convergents.† Thus,

$$\sqrt{14}=3+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{6+}\dots,$$

and $c=4$. Our first approximation should be taken as $\frac{15}{4}$, which is the fourth convergent. The iteration gives $\frac{449}{120}$, which is the eighth convergent, and so on. If we had taken $\frac{11}{3}$, which is the third convergent, the iteration gives $\frac{247}{66}$, which is a poorer approximation than the simpler fraction $\frac{116}{31}$, the sixth convergent. This would seem to indicate the desirability of proceeding as we have indicated.

The necessary modification to cover square roots of fractions is easily made. The problem can be reduced to the evaluation of $\sqrt{N}/M$, and our formula still holds for the continued fraction expansion of $\sqrt{N}/M$, provided N be replaced by N/M^2.

Raymond Garver.

* Chrystal, *Algebra*, Part II, 2nd edition, 1926, 468.

† When $c=2$, but not for larger values, it is still true that any convergent iterates to give a convergent.

1053. *Application of a useful lemma to the series for* $\sin\alpha$ *and* $\cos\alpha$.

The Lemma, given * in Note 1028, is here used in its simplest form, that if $u_1, u_2, \ldots$ are positive numbers each less than unity, then

$$(1-u_1)(1-u_2)(1-u_3)\ldots > 1-u_1-u_2-u_3-\ldots.$$

We have

$$\sin n\theta = n\sin\theta\cos^{n-1}\theta - \frac{n(n-1)(n-2)}{3!}\sin^3\theta\cos^{n-3}\theta + \ldots.$$

Putting $n\theta=\alpha$ and supposing n a large integer and θ small, we may write

$$\sin\alpha = \alpha\left(\frac{\sin\theta}{\theta}\right)\cos^{n-1}\theta - \frac{\alpha^3}{3!}\left(\frac{\sin\theta}{\theta}\right)^3\cos^{n-3}\theta\,.\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots \quad (1)$$

Let

$$S' = \alpha\left(\frac{\sin\theta}{\theta}\right)\cos^{n-1}\theta - \frac{\alpha^3}{3!}\left(\frac{\sin\theta}{\theta}\right)^3\cos^{n-3}\theta + \ldots \quad (2)$$

to the same number of terms.

$$S = \alpha - \frac{\alpha^3}{3!} + \ldots \quad (3)$$

to the same number of terms.

Now if

$$V = v_1 - v_2 + v_3 - v_4 \cdots,$$
$$W = w_1 - w_2 + w_3 - w_4 \cdots,$$

where the v's and w's are positive numbers and each v is less than the corresponding w, then

$$W - V = (w_1-v_1)-(w_2-v_2)+(w_3-v_3)-\ldots,$$
$$|W-V| < (w_1-v_1)+(w_2-v_2)+(w_3-v_3)+\ldots.$$

Applying this to (1) and (2) we have

$$|S'-\sin\alpha| < \sum\frac{\alpha^{2r-1}}{(2r-1)!}\left(\frac{\sin\theta}{\theta}\right)^{2r-1}(\cos\theta)^{n-2r+1}$$
$$\times\left[1-\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{2r-2}{n}\right)\right].$$

But by the lemma the last bracket is less than

$$\{1+2+\ldots+(2r-2)\}/n = (2r-1)(2r-2)/2n.$$

Also $\sin\theta/\theta$ and $\cos\theta$ are each less than unity, so that

$$|S'-\sin\alpha| < \frac{1}{2n}\sum\frac{\alpha^{2r-1}}{(2r-3)!} < \frac{\alpha^2}{2n}\,.\,e^{\alpha}. \quad (A)$$

Moreover, since $\tan\theta>\theta$, $\sin\theta/\theta>\cos\theta$, therefore by (2) and (3)

$$|S-S'| < \sum\frac{\alpha^{2r-1}}{(2r-1)!}\left[1-\left(\frac{\sin\theta}{\theta}\right)^{2r-1}(\cos\theta)^{n-2r+1}\right]$$
$$< \sum\frac{\alpha^{2r-1}}{(2r-1)!}(1-\cos^n\theta).$$

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* *Mathematical Gazette*, XVI, No. 218, 132 (1932).

But $1-\cos^n\theta=(1-\cos\theta)(1+\cos\theta+\cos^2\theta+\ldots+\cos^{n-1}\theta)$

$$<n(1-\cos\theta)=2n\sin^2\tfrac{1}{2}\theta$$

$$<2n\,.\,\tfrac{1}{4}\theta^2=\alpha^2/2n.$$

Hence $$|S-S'|<\frac{\alpha^2}{2n}\sum\frac{\alpha^{2r-1}}{(2r-1)!}<\frac{\alpha^2}{2n}e^{\alpha}. \quad \text{.............................(B)}$$

By (A) and (B) $$|S-\sin\alpha|<\frac{\alpha^2}{n}e^{\alpha},$$

which tends to zero as n tends to infinity. Thus

$$\sin\alpha=\lim_{n\to\infty}S=\alpha-\frac{\alpha^3}{3!}+\frac{\alpha^5}{5!}-\ldots \textit{ to infinity.}$$

Similarly for $\cos\alpha$.

PERCY J. HEAWOOD.

1054. *A method for division of decimals.*

For the division, say, of 1·7384 by 6·78, the divisor is 678 hundredths, hence the integral part of the quotient ends when the hundredths in the dividend are used.

```
          0|25
6·78 ) 1·73 84
       1 35 6
       ------
         38 24
         33 90
         -----
         ·04 34
```

Consequently the rule :

The divisor is in hundredths; draw a vertical line after the hundredths in the dividend to mark the decimal point in the quotient, and divide through. The remainder after two decimal places in the quotient is read off immediately as ·0434.

The method follows division by integers readily. Some preliminary practice in reading off is needed by beginners; for example, 4·7 is 47 tenths. The method is suited to all combinations of divisor and dividend, and has been found successful.

H. A. BAXTER.

1055. *The operation* $\{f(D)\}^{-1}e^{mx}$.

In the example given by Mr. H. V. Lowry * in Note 1027, on the method of finding the particular integral of the differential equation

$$(D^2+2D-3)y=\sinh x,$$

it may be worth noting that the error there mentioned is really due to the presence of the factor $(D-1)$ in the operator. For when $\sinh x$ is written as $\frac{1}{2}(e^x-e^{-x})$, the ordinary method is " safe " for

* H. V. Lowry, *Math. Gazette*, XVI, 131 (1932).

e^{-x}, but leads to the case of failure for e^x. Hence the particular integral is

$$\tfrac{1}{2}\{(D-1)(D+3)\}^{-1}(e^x-e^{-x})$$
$$=\tfrac{1}{8}(D-1)^{-1}e^x+\tfrac{1}{8}e^{-x}$$
$$=\tfrac{1}{8}(xe^x+e^{-x}).$$

This caution is always necessary when an operator containing a factor $(D\pm m)^{-k}$ acts upon $\sinh mx$ or $\cosh mx$, or when a factor $(D\pm im)^{-k}$ acts upon $\sin mx$ or $\cos mx$. F. UNDERWOOD.

University College, Nottingham.

1056. *The teaching of differentials.*

Two articles on this subject have appeared in recent numbers * of the *Gazette*. They left some doubt as to the significance of the equation

$$\Delta y = A\Delta x + \epsilon\Delta x,$$

perhaps because the writers concentrated entirely on the logical side of differentials, not keeping in mind the way in which they are used in deriving the differential equations of mathematical physics. May I suggest that the right point of view is as follows:—

There is no distinction between "having a derivative" and "being differentiable" (any doubt here may have arisen through de la Vallée Poussin not being as transparently clear as usual on this point). A function $y=f(x)$ is said to have a derivative or to be differentiable at x if the fraction

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

has a limit when $\Delta x\to 0$. The limit is then denoted by $f'(x)$, and the product $f'(x)\Delta x$ is called the differential of y with respect to x and is denoted by dy. It follows that

$$\Delta y = f'(x)\,.\,\Delta(x)+\epsilon\Delta x,$$

or

$$\Delta y = dy+\epsilon\Delta x,$$

where $\epsilon\to 0$ when $\Delta x\to 0$.

Conversely, it follows that, if

$$\Delta y = A\Delta x+\epsilon\Delta x, \quad\text{..................................(1)}$$

where $\epsilon\to 0$ when $\Delta x\to 0$, and A is independent of Δx, then $A\Delta x$ is the differential of y, or $A=f'(x)$, if $y=f(x)$.

The point that the writers of the articles seem to miss is that the converse is worth mentioning because it commonly occurs in the applications of the calculus to geometrical and physical problems. A great many of these applications have been successful because they have been discovered by deep insight into what happens during infinitesimal changes in the variables concerned, leading naturally to the notation of differentials and equations typified by (1), and hence to differential equations typified by

$$\frac{dy}{dx}=A,$$

* July, 1931; February, 1932.

from which y is to be found. In other words, the converse needs emphasising because from such an equation as (1) the differential coefficient dy/dx is found *before the function y is known*: (in the case of more than one independent variable, partial differential equations are obtained by similar reasoning).

As a trivial illustration, let us *find the area under the curve $y=f(x)$*. The argument which is typical of physical applications would be as follows: if z is the area between the ordinates at a and x, and if $f(x)$ is monotonic between x and $x+\Delta x$, then

$$\Delta z \text{ lies between } y\Delta x \text{ and } (y+\Delta y)\Delta x,$$

and hence, if y is continuous,

$$\begin{aligned}\Delta z &= (y+\theta\Delta y)\Delta x \\ &= y\Delta x + \theta\Delta y\Delta x, \quad (0<\theta<1).\end{aligned}$$

Here $\epsilon=\theta\Delta y$, and hence $\epsilon\to 0$ when $\Delta x\to 0$, and we deduce

$$\frac{dz}{dx}=y.$$

Thus dz/dx is found before z is known.

Often the term in ϵ is regarded as intuitively ignorable in physical applications of the argument. F. BOWMAN.

1057. *Green's Theorem.*

The following proof of Green's theorem should prove both useful and instructive to physicists, as it appears to be simpler and more direct than the usual proof. I have been unable to find mention of it in any text-book.

Let V be any finite continuous uniform (scalar) point function.

Consider an element of volume dv, $ABCDA'B'C'D'$, enclosed between two adjacent equipotential surfaces $ABCD$, $A'B'C'D'$, of

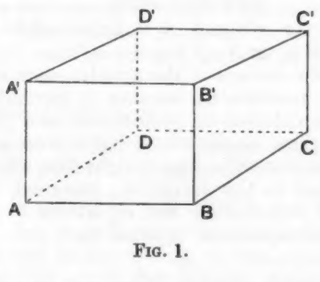

FIG. 1.

V, such that AA', BB', CC', DD', are normals to the surfaces. Let $ABCD$ be a rectangle of sides $AB=dx$, $AD=dy$, and let $AA'=dz$.

Then if n is the outward drawn unit normal vector to an elementary area dS of surface, we have

$$\int n V dS = -V_A . AB . AD . m + V_{A'} . AB . AD . m$$

where the integral is taken over the surface of dv; m is the outward drawn unit normal vector to the equipotential surface at A; and V_A, $V_{A'}$ are the values of V corresponding to the points A and A'.

Thus $$\int n V dS = (V_{A'} - V_A) . AB . AD . m$$
$$= \frac{\partial V}{\partial s} . m . dv,$$

if AA' is the direction s.

In other words, $$\int n V dS = \nabla V dv,$$

since AA' is normal to an equipotential surface of V.

Now any volume inside a closed surface can be divided up into small elements such as dv bounded by equipotential surfaces. The effect at the closed surface is shown at P in the diagram, which is self-explanatory.

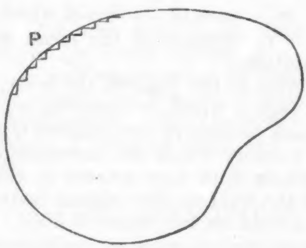

FIG. 2.

Hence for any closed surface $\int n V dS = \int \nabla V dv$, for as with the elementary line integrals in the proof of Stokes's theorem, the surface integrals over the small elements cancel one another, leaving the integral over the whole outside surface.

Hence we obtain three equations of the form

$$\int n . i V_1 dS = \int i . \nabla V_1 dv$$
$$= \int \operatorname{div} (i V_1) dv,$$

which on adding give

$$\int n . F dS = \int \operatorname{div} F dv$$

where F is any finite continuous vector point function.

G. F. LEWIN.

1058. *A problem on " Centrifugal Force ".*

The problem is to investigate the behaviour of a body immersed in liquid which rotates about a vertical axis.

Experiment. A ball is placed in a test-tube containing water. The test-tube is inclined at an angle α to the horizontal and is rotated with uniform angular velocity ω about a vertical axis.

A curious phenomenon is then observed. It is found that if ω is sufficiently great a cork ball sinks " like lead ", and a leaden ball floats " like a cork ".

A convenient arrangement of the apparatus is shown in the diagram below.

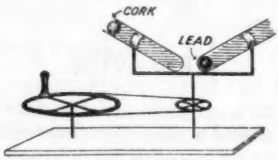

FIG. 1.

Explanation. Let m = mass of immersed body,

m' = mass of the liquid which it displaces,

r = distance of the mean centre of the body from the axis of rotation.

To discuss the action of the fluid on the body, suppose the body removed, and the region which it occupied replaced by an equal volume of liquid, each particle of the replaced liquid being supposed endowed with the velocity which the corresponding particle of the body had. The whole fluid now rotates in relative equilibrium. Now, the action of the fluid on the original body is the same as the action of that same fluid on the replaced fluid. And this replaced fluid is in equilibrium (that is, relative equilibrium) under

(1) this action,

(2) the weight $m'g$ vertically down through the mean centre,

(3) the centrifugal force $m'r\omega^2$ through the mean centre.

Thus for equilibrium, the action of the rest of the fluid on the replaced fluid must have two components :

(1) $m'g$ vertically upwards,

(2) $m'r\omega^2$ horizontally inwards,

both acting through the mean centre of the body. And, as we have seen, these also represent the action of the fluid on the original body m.

Consider now the forces acting on m (see Fig. 2). They are

(1) $m'g$ vertically upwards,

(2) $m'r\omega^2$ horizontally towards the axis,

(3) mg vertically downwards,

(4) $mr\omega^2$ horizontally outwards from the axis.

These all act through the mean centre of the body.

In addition there will possibly be a reaction R if and when the immersed body touches the walls of the tube. Since the contact will be practically frictionless, the direction of R will be normal to the walls of the tube. It may therefore be eliminated from consideration by resolving the forces parallel to the axis of the tube.

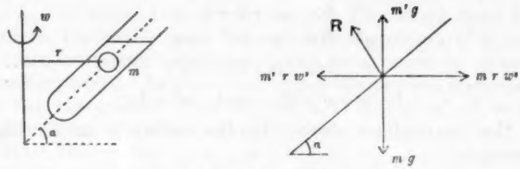

FIG. 2.

We thus have that the resultant force, F say, *down* the tube, acting on the immersed body is given by the equation

$$\begin{aligned} F &= (m - m')g\sin\alpha - (m - m')r\omega^2\cos\alpha \\ &= (m - m')(g\sin\alpha - r\omega^2\cos\alpha). \end{aligned}$$

Case 1. $m > m'$. The body sinks in the fluid at rest.

F is clearly positive for small values of ω, that is, for small speeds of rotation. Hence, the particle to begin with stays at the foot of the tube. As ω increases F becomes negative, and therefore the particle travels up the tube. The quantity r increases and will continue to do so since F remains negative. The body therefore comes to the surface.

Case 2. $m < m'$. The body floats in the fluid at rest.

$$F = (m' - m)(r\omega^2\cos\alpha - g\sin\alpha).$$

For small values of ω, F is negative, that is, the resultant force is directed up the tube : the body behaves normally.

But for larger values of ω, F is positive : the force is directed downwards and the body sinks. The quantity r therefore decreases and will reach a value where $F = 0$. In this position there is relative equilibrium. If ω is sufficiently large the body will go right to the foot of the tube.

These results are easily demonstrated experimentally : the leaden ball is either definitely at the foot of the tube or else floating, whereas the cork ball, by adjusting the speed of rotation, can be made to occupy any intermediate position besides the two extreme ones.

The speeds necessary to exhibit these facts are quite small. The phenomenon does not appear to be well known, and it is thought that both the experiment and the simple explanation of the results given above should prove to be of interest to teachers and students of elementary Mechanics.

W. G. GUTHRIE.

1059. *On Note* 1042.

To solve

$$\begin{array}{c|c|c|ll} 1 & 1 & 1 & x^2+2yz=a, & \text{.....................(1)} \\ \omega & \omega^2 & 1 & y^2+2zx=b, & \text{.....................(2)} \\ \omega^2 & \omega & 1 & z^2+2xy=c. & \text{.....................(3)} \end{array}$$

Write $\quad p^2$ for $a+b+c$,

$\quad q^2$ for $a+b\omega^2+c\omega$,

$\quad r^2$ for $a+b\omega+c\omega^2$,

where $\quad 1+\omega+\omega^2=0$, and $\omega^3=1$.

Using the multipliers shown in the columns and adding the results, we get

$$\Sigma x^2+2\Sigma yz=a+b+c;$$

that is, $\quad x+y+z=\pm p$, ...(4)

and $\quad x+\omega y+\omega^2 z=\pm q$, ...(5)

and $\quad x+\omega^2 y+\omega z=\pm r$. ....(6)

By addition of (4), (5) and (6) we get

$$3x=\pm p\pm q\pm r;$$

thus $\quad x=\pm\frac{1}{3}\{\surd(a+b+c)\pm\surd(a+b\omega^2+c\omega)\pm\surd(a+b\omega+c\omega^2)\};$

y and z may be written down by cyclic mutation, or by using the same multipliers for (4), (5) and (6) as for (1), (2) and (3) and adding the results. F. C. Boon.

1060. *On Note* 1044.

In Note 1044 (*Math. Gazette*, October 1932, pp. 267-8) Mr. Forder gives a proof that the arithmetic mean is not less than the geometric mean.

Mr. F. P. White writes: " This proof is given in Todhunter, *Algebra* (§ 681 of the 1879 edition), at any rate in essence. But on the whole matter see Enriques, *Questioni riguardanti le matematiche elementari*, 3rd edition, not dated but 1927, parte terza, pp. 124-134, article by A. Padoa. He gives the reference Cauchy, *Œuvres*, ser. 2, t. 3, pp. 375-377 ".

Mr. E. G. Phillips sends the Cauchy reference, and also points out that the proof is given in his *Course of Analysis* (Cambridge, 1930), p. 144.

1061. *The Numerical Evaluation of a Derivative.*

If in the familiar result

$$\lim_{h\to 0}\frac{u(x+h)-u(x-h)}{2h}=u'(x)$$

we substitute ky for h we have

$$ku'(x)=\lim_{y\to 0}\frac{u(x+ky)-u(x-ky)}{2y}.$$

If then for some value of k we can tabulate the function

$$\{u(x+ky)-u(x-ky)\}/2y$$

as a function $v(y)$ of y for a sufficient number of values of y, interpolation into this table for the value $y=0$ is a means of computing $u'(x)$.

Suppose $u(x)$ tabulated for a succession of values at a constant interval h. In order that $x+ky$ and $x-ky$ should both be among these values, their difference $2ky$ must be a multiple of h, and therefore x must be either a tabular point or a point midway between two tabular points. In these two cases the process indicated, modified by the recognition that $v(y)$ is a function of y^2, is in fact a practical process of extreme rapidity.

(1) Given the values $\dots u_{-2}, u_{-1}, u_0, u_1, u_2, \dots$, the value of hu_0' is the value at $t=0$ of the function whose values for $t=1, 4, 9, \dots$ are $\frac{1}{2}(u_1-u_{-1})$, $\frac{1}{4}(u_2-u_{-2})$, $\frac{1}{6}(u_3-u_{-3})$, $\dots$.

(2) Given the values of $u(x)$ for $x=\dots a-2h, a-h, a, b, b+h, b+2h, \dots$, where $h=b-a$, the value of $hu'\{\frac{1}{2}(a+b)\}$ is the value at $t=0$ of the function whose values for $t=1, 9, 25, \dots$ are $u(b)-u(a)$, $\frac{1}{3}\{u(b+h)-u(a-h)\}$, $\frac{1}{5}\{u(b+2h)-u(a-2h)\}$, $\dots$.

It is of course perfectly easy to deduce explicit formulae for approximations to the derivative in these cases, but to regard this as an advantage from the computer's standpoint is a complete misapprehension.

A process for evaluating $u'(x)$ only at specified points is effective for the tabulation of this function. For the computation of $u'(x)$ at an arbitrary isolated point X, which divides the interval from a to b into the parts θh, ϕh, such a process is insufficient. Two solutions are obvious.

(1) We may compute a sufficient number of values of $u'(x)$ at tabular points and midpoints to determine $u'(X)$ by interpolation.

(2) We may compute $u(X)$ by interpolation. We can then tabulate the function $\{u(a+hy)-u(X)\}/(y-\theta)$, and interpolation for $y=\theta$ gives the value of $hu'(X)$.

The object of this Note is to point out a third solution. The product $hu'(X)$ is the limit of the quotient

$$\frac{u(b+hx)-u(a-hy)}{y+z+1},$$

as $y\to-\theta$, $z\to-\phi$. To substitute the limiting value of either variable requires the calculation of $u(X)$. But integral values of y and z involve only tabular values of $u(x)$. We can therefore tabulate the function of two variables immediately, for as large a range of integral values as we wish, and bivariate interpolation into this easily constructed table gives the derivative directly. As in the two univariate cases, there are features of symmetry which simplify the interpolation. E. H. N.

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896. "Every day I read, I am more and more thoroughly convinced of my incapacity for the subject [mathematics]."—W. E. Gladstone's diary for 1831, quoted in J. Morley, *Life of Gladstone* (1905), ii. 79.

CORRESPONDENCE.

REPLY TO "A CHALLENGE".

To the Editor of the *Mathematical Gazette.*

DEAR EDITOR,—This reply to "Wrangler's" challenge in the current number of the *Gazette* is sent off post-haste, as I believe that the acceptors of such challenges strove to be first in the field.

To find p the probability that a man of age a reaches age b, multiply $(b-a)$ by the mean of the reciprocals of the expectations of life from a to b, add the logarithm of the expectation at b and subtract the logarithm of the expectation at a, thus obtaining the logarithm of $1/p$ (natural logarithms are to be used or the product should be multiplied by μ).

If $f(x)$ is the chance that a man of age a reaches the age $a+x$, and $\phi(x)$ the expectation of life at that age,

$$\phi(x)=\frac{1}{f(x)}\int_x^\infty f(x)dx.$$

or if $f(x)$ is the derivative of a function $F(x)$,

$$\frac{1}{\phi(x)}=\frac{f(x)}{F(\infty)-F(x)}, \quad\text{..............................(i)}$$

and, integrating, we have

$$\int_0^x \frac{dx}{\phi(x)}=-\log\frac{F(\infty)-F(x)}{c},$$

where c is a constant. Thus making use of (i),

$$\int_0^x \frac{dx}{\phi(x)}=-\log\frac{f(x)\,.\,\phi(x)}{c}.$$

As $f(0)=1$, we have $c=\phi(0)$ and

$$\log f(x)=\log\phi(0)-\log\phi(x)-\int_0^x\frac{dx}{\phi(x)}.$$

For example, if the expectations of life at yearly intervals from 50 to 60 are 20·3, 19·5, 18·9, 18·2, 17·6, 16·9, 16·2, 15·6, 15·0, 14·4, 13·8, then

$$\log\phi(0)-\log\phi(x)=\log 20{\cdot}3-\log 13{\cdot}8=\cdot 7080-\cdot 3221,$$

while for the integral Simson's rule gives ·5979, so that $\log p=\bar{1}{\cdot}7880$ and $p=\frac{4}{5}$ approximately. Consequently the chance that two men of 50 reach 60 is about $\frac{16}{25}$, that neither do so about $\frac{1}{25}$, and that one only does so about $\frac{8}{25}$.

C. H. HARDINGHAM.

July 3, 1932.

IS THE EARTH ROUND OR FLAT?

To the Editor of the *Mathematical Gazette.*

SIR,—Has the above question any meaning? If it is not possible for human beings to prove that the Earth is either round or flat, surely the question becomes meaningless. I give below reasons for thinking that we cannot answer the question one way or the other.

Let us take a system of three unit vectors, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, at right angles to each other and use spherical polar coordinates, viz. ϕ for the co-latitude measured from $\mathbf{e}_3$, θ for the meridian angle measured from $\mathbf{e}_1$, r for the radius vector.

The differential vector $d\mathbf{r}$ of Euclidean 3-space using these coordinates is

$$(1)\quad d\mathbf{r}=r(\cos\phi\cos\theta\,.\,\mathbf{e}_1+\cos\phi\sin\theta\,.\,\mathbf{e}_2-\sin\phi\,.\,\mathbf{e}_3)d\phi$$
$$+r(-\sin\phi\sin\theta\,.\,\mathbf{e}_1+\sin\phi\cos\theta\,.\,\mathbf{e}_2)d\theta$$
$$+(\sin\phi\cos\theta\,.\,\mathbf{e}_1+\sin\phi\sin\theta\,.\,\mathbf{e}_2+\cos\phi\,.\,\mathbf{e}_3)dr.$$

Squaring (1) we get for the square of the line element (or ground form)

$$(2)\quad ds^2=(d\mathbf{r})^2$$
$$=r^2\,d\phi^2+r^2\sin^2\phi\,d\theta^2+dr^2.$$

Putting $r=a$ in (1) we get for the differential vector of a sphere of radius a, in 3-space,

$$(3)\quad d\mathbf{r}=a(\cos\phi\cos\theta\,.\,\mathbf{e}_1+\cos\phi\sin\theta\,.\,\mathbf{e}_2-\sin\phi\,.\,\mathbf{e}_3)\,d\phi$$
$$+a(-\sin\phi\sin\theta\,.\,\mathbf{e}_1+\sin\phi\cos\theta\,.\,\mathbf{e}_2)d\theta,$$

with ground form

$$(4)\quad ds^2=a^2\,d\phi^2+a^2\sin^2\phi\,d\theta^2.$$

Next consider the non-Euclidean 3-space whose differential vector is, with ϕ, θ and r as parameters,

$$(5)\quad d\sigma=r\,.\,\mathbf{e}_1\,d\phi+r\sin\phi\,.\,\mathbf{e}_2\,d\theta+\mathbf{e}_3\,.\,dr.$$

Squaring it, we get its ground form :

$$(6)\quad ds^2=r^2\,d\phi^2+r^2\sin^2\phi\,d\theta^2+dr^2.$$

Consider the Riemannian 2-pole elliptic plane with constant $\frac{1}{a}$, lying in this non-Euclidean 3-space. It is obtained by putting $r=a$ in (5). Its differential vector is

$$(7)\quad d\sigma=a\,.\,\mathbf{e}_1\,d\phi+a\sin\phi\,.\,\mathbf{e}_2\,d\theta.$$

Its ground form is

$$(8)\quad ds^2=(d\sigma)^2$$
$$=a^2\,d\phi^2+a^2\sin^2\phi\,d\theta^2.$$

By comparing their ground forms (2) and (6), we see that the Riemannian 3-space is " applicable " to Euclidean 3-space.

By comparing (4) and (8) we see that the Riemannian plane is " applicable " to the Euclidean sphere. Let us now suppose that two persons E and N move about the Earth in company with each other. Any measurements they may make will be the same, e.g. if they measure the sides and angles of a geodesic triangle, they will get the same relations connecting the sides and angles as given in spherical trigonometry. E chooses to interpret such measurements as proving that the surface is a sphere of radius a, lying in Euclidean 3-space. N chooses to interpret them as proving that the surface is the above-mentioned Riemann plane lying in the non-Euclidean 3-space (5). The geometries of these surfaces and spaces are the same. Therefore no possible experiment can decide between them.

The proofs given in books on geography and astronomy beg the question by assuming our 3-space Euclidean. A corresponding argument applies to the case of a spheroid. A. W. King.

Imperial College of Science and Technology,
10th June, 1932.

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897. (Madame du Châtelet) was intellectual and sensuous—an agreeable blend. She liked books, diamonds, algebra, petticoats, and physics. In this she was like Voltaire. . . .—A. Maurois, *Voltaire*, p. 53. [Per Mr. E. H. Lockwood.]

REVIEWS.

Topology. By S. Lefschetz. Pp. x, 410. $4.50. 1930. American Math. Soc. Colloquium Publications, 12. (American Math. Soc.)

Analysis Situs. By O. Veblen. Second Edition. Pp. x, 194. $2.00. 1931. American Math. Soc. Colloquium Publications, 5. (American Math. Soc.)

Topology, which first emerged as a collection of curious problems on crossing bridges and colouring maps, was lifted from the limbo of mathematical recreations into the domain of serious mathematics by the part given it by Riemann in his Theory of Algebraic Functions. His topology, though convincing, was founded on the crudest intuitional reasoning, and Weierstrass made his theory independent of it, considering it unsuitable for the building up of a rigorous theory. Even Poincaré, whose work of 1895-1910 fixed the main lines of the " combinatorial " side of the subject, was little interested in a logically perfect development of the subject, and though he defined his spaces more carefully than Riemann, his proofs can hardly be accepted to-day as complete. For a rigorous treatment it was necessary to import the ideas of point-set theory, and especially the theory of single-valued, but not necessarily (1, 1), continuous transformations, first systematically investigated by Brouwer. The use of these non-invertible mappings in defining the homology characters (Betti numbers, etc.) completely transformed the subject; the intractable topological mappings, with their interminable approximation lemmas, gave place to simple deformation processes, and it was possible to give not only a rigorous but a readable presentation of the subject. This important innovation was made by Veblen and Alexander, and the first systematic account of the simplified theory was contained in the first edition (1922) of Veblen's *Analysis Situs*.

This book is the familiar companion of all workers in the field, and it is hardly necessary to recommend it further. It is a particularly clear presentation of topology up to 1922, and in reprinting it its author has wisely decided not to alter it substantially, leaving to others the task of describing what has happened since its first publication. The most important alteration is the clearing up of the algebraic part of the invariance proof for the Betti numbers and torsions, which was not quite satisfactory in the first edition.

As late as 1925 it was possible for the writer of a widely-used text-book to refer to topology as the " Sorglingkind der Mathematik ", which might if encouraged be able subsequently to render some aid to its more robust relations. Since the discovery of the fixed-point formula of Lefschetz it can no longer be doubted that here is an instrument of great power for resolving problems in a variety of branches of mathematics: Algebraic Geometry and the Calculus of Variations are two of the domains in which it has already been employed with decisive effect. The Lefschetz formula lies deep—it uses all the resources of Poincaré's theory of dual bases of homology, and of the " Kronecker index " of two intersecting complexes, as further developed by Lefschetz himself; and its proof forms a fitting climax to his book.

This account differs from Veblen's in striving from the start for the greatest possible generality. All the relations are proved, as far as possible, " mod L ", that is, neglecting the cells of a certain arbitrary sub-complex L, with a view to subsequent applications to infinite complexes; and " mod m ", that is, reducing all the coefficients of the homologies with respect to an arbitrary modulus m; and it must be admitted that the account would make stiff reading as a first introduction to the subject. The existence of Veblen's book, however, makes it unnecessary for Lefschetz to attempt the difficult task of being both elementary and general, and his pages contain an extraordinary wealth of theorems both proved and adumbrated.

The plan of the book is as follows. In the first chapter the purely com-

binatorial part of the theory of homologies is developed and the Betti numbers and torsions defined. Chapter II is occupied with the invariance proofs when general " singular " cycles are allowed. Next the duality properties of manifolds are investigated, and a remarkable theorem unifying the duality theorems of Poincaré and Alexander is proved. The next two chapters, on intersections of complexes and on product complexes, prepare the way for Chapter VI, in which the central results, on fixed points of transformations of manifolds, are obtained. Chapter VII is on a subject that has greatly occupied topologists recently—an extension to general compact spaces of the combinatorial theorems about manifolds. Chapter VIII contains some applications to algebraic varieties (the subject of a separate book of the author's, published some years ago in the *Collection Borel*), and the book is terminated by a particularly full bibliography.

The proofs of many of the theorems are difficult, and sometimes leave a good deal to the reader, but this ruggedness of treatment is very much more stimulating than the glassy polish of a " classical " treatment which convinces the reader that all is over. This is a quite indispensable book for anyone who proposes to apply the fixed-point theory in any of the many domains that seem to invite its use. M. H. A. N.

Modular Invariants. By D. E. Rutherford. Pp. viii, 84. 6s. 1932. Cambridge Tracts, 27. (Cambridge University Press)

The ordinary algebraic theory of invariants has been a familiar and productive branch of mathematics for a great number of years; the modular theory, on the other hand, is a product of the present century alone.

In this volume of the Cambridge Tracts, Rutherford performs a very useful service in placing before the mathematical public a clear and concise account of the elements of the theory as at present developed. His acknowledgments in the preface to Professor Turnbull and Professor Weitzenböck for their help in reading proofsheets and making suggestions, are a sufficient guarantee, if such were needed, of the excellence of his account.

The adjective " modular " indicates that congruences are to take the place of algebraic equations, and that those numbers which appear may no longer be just ordinary real or complex numbers, but, instead, members of the Galois Field, G.F. $[p^n]$. There are, however, different possibilities. Practically all algebraic symbols, at least in their origin, represent number. In the theory of invariants there are three different classes of symbols either actually present or else implied. There are first the coefficients a of the ground-forms, there are secondly the variables, and thirdly the coefficients $\mathbf{a}$ of transformation. In addition number appears explicitly in the form of integers multiplying the separate products of symbols in the integral algebraic functions which are the concomitants. The term modular may be supposed to apply to these last numbers, and not to the symbols themselves; or else it may also be applied to one or other of the classes of symbols, or finally to all. And, indeed, different writers have adopted different points of view. Rutherford therefore divides the subject of invariants into five types. The first is the classic theory in which both symbols and explicit integers belong to the C.F. (the Complex Field). The remaining four types belong to the theory of modular invariants; in all of them the congruence $p \,|||\, 0$ (to use the notation of the Tract) is applicable to the explicit integers. In the second type, " Congruent concomitants ", the various classes of symbols are all treated as in the ordinary theory. In the third type, " Formal Concomitants ", the ground-form coefficients a belong to the C.F., but the coefficients of transformation a belong to G.F. $[p^n]$, and in consequence Fermat's theorem, $\mathbf{a}^{p^n} \,||\, \mathbf{a}$, may be used. In the fourth type, " Non-formal Concomitants ", the position as to coefficients of ground-forms and those of transformation is reversed; so far no one has under-

taken the discussion of concomitants of this type. In the fifth type, "Residual Concomitants", both sets of coefficients belong to G.F. $[p^n]$.

There is clearly a connection between concomitants of different types; *e.g.* all those of either of the first two types are concomitants also of the later types, but not usually an exhaustive set, and many of them become congruent to zero. There is a similar connection between the concomitants of the third and fourth types and those of the last type. In relation to this the author gives Miss Sanderson's interesting theorem on the connection between invariants of the third and the last types.

The author treats each type in succession, giving most attention to the last, in particular to the theory of Characteristic Invariants due to Dickson.

In the case of algebraic invariants, the problem of the "Finiteness" has always occupied a central position; and here the second part of the Tract works up to the Finiteness Theorem for Modular Covariants due to E. Noether.

In the course of the work the question of the application of a symbolical calculus similar to that applied so successfully by Aronhold to the ordinary algebraic theory arises. The method employed is due originally to Sanderson, and was developed later by Hazlitt. In the modular theory a difficulty at once presents itself, owing to the presence of the binomial coefficients which appear when a ground-form is represented as a power of a linear symbolical form. This difficulty is overcome by writing the ground-form of order m as the product of m symbolical linear factors. In a concomitant of order δ each symbol must appear δ times, and the m different symbols must appear symmetrically. The method corresponds to the representation of ordinary covariants in terms of differences of the roots and variables, rather than to the symbolical calculus; and, indeed, in her presentation of the matter, Sanderson regards the symbols as Galois imaginaries in a higher field. This fact makes it clear that the method in its present form is only applicable to binary forms, and in fact would lead to the vanishing of many concomitants of ternary forms, or forms of more variables.

A review, however condensed, of the present state of knowledge in any subject is a very valuable thing, but a difficult task to undertake. Dickson in his *History of the Theory of Numbers*, vol. 3, ch. 19, gave such a summary on the subject of modular invariants up to the end of 1921. Rutherford in an appendix to this Tract carries this on practically up to date. In a second appendix he gives a list of the papers which have appeared on the subject.

ALFRED YOUNG.

The Foundations of Differential Geometry. By O. VEBLEN and J. H. C. WHITEHEAD. Pp. x, 98. 6s. 6d. 1932. Cambridge Tracts, 29. (Cambridge University Press)

When an appreciation of the discoveries of Bolyai and Lobachewski rendered inevitable an extension of the meaning of geometry, the foundations of a geometry took the form of a body of axioms, of which the most important are incidence relations concerning points, recognised as indefinable elements, and certain equally indefinable classes of these elements. A considerable amount of research on these foundations dealt with the introduction of coordinates, but the development of this research did not modify an instinctive realisation that a geometry can be based directly on a coordinate system; the validity of a coordinate geometry is not contingent on the proof that it is in fact a form of geometry implicit in some set of axioms already canonised. But a coordinate geometry has to overcome an initial difficulty which has no analogue in pure geometry: the frame of reference has as a rule no unique significance. The coordinates of a point depend no less on the frame than on the point itself, and in many cases any point may have any set of coordinates; the point is identified not by its coordinates simply, but by its coordinates in a

frame in some way specified, and we recognise the identity of a point by knowing the relations between coordinates in one frame and coordinates in another. Thus we come to Klein's conception of a geometry as the theory of invariance under a group of transformations.

This conception systematised the field to which it was applied, but while it prevented the name of geometry from covering many logical investigations for which mathematicians are agreed that it is unsuitable, it threatened also to distort the study of infinitesimal geometry. To Gauss, to Lamé, and later to Darboux, curvilinear coordinates were parameters in terms of which cartesian coordinates could be expressed. From this point of view, no question of foundations arises, except for the space to which the rectilinear coordinates belong. But Riemann, as early as 1854, asserted that infinitesimal geometry, the geometry of continuous manifolds, should depend only on curvilinear coordinates within the manifold; the quadratic forms, and indeed any analytical magnitudes whatever used in the specification, should be expressed from the first in terms of these coordinates. The relations between the coefficients of the fundamental quadratic forms of a surface in euclidean space are not essential to the notion of a two-dimensional manifold, and the study of such a manifold is a geometry, whether or not it is the geometry of any surface in the classical sense. Klein's conception does not cover this view of a geometry as the theory of a set of points with a definable structure. It is true that, if we can prove that any two-dimensional riemannian manifold must exist as a surface in a euclidean space of sufficiently high order, though not necessarily in one of order three, we shall in a sense have brought the study of such a manifold within the range of geometry otherwise conceived, but we shall have evaded rather than solved the problem of the intrinsic status of the manifold. Relations between one manifold and another in which it is embedded are a proper subject of study, but that a manifold should be intelligible only in virtue of being embedded in another of a particular type is repugnant to the pure mathematician; to the applied mathematician, anxious to understand the structure of the four-dimensional space-time world, the suggestion that a fifth or sixth dimension is logically implied by the very questions he is asking is absurd.

Thus comes the need for an enlarged definition of a geometry, and for a critical account of the assumptions that lead to geometries specified by their differential structure. The collaboration of Professor Veblen and Mr. Whitehead is a natural and a happy one. The childish figures are perhaps a blemish on their pages. To suppose that anyone contemplating the reading of a book on this subject could need instruction in the elements of the theory of linear dependence or be unacquainted with the definition of a group is a sad reflection on our universities. The restriction to real coordinates is much more serious than a casual footnote on the first page would lead the unwary to expect; with the real number-system non-singular transformations fall automatically into the two classes of positive and negative, but with complex numbers there is no such natural division and the problem of orientation is complicated; nevertheless, even elementary differential geometry, with its minimal lines and its tetracyclic plane, uses complex coordinates as a matter of course. Except in the last respect, the scope of this Tract is wider than the title, for the whole problem of the nature of geometry is shown as a historical and logical setting to the problem of the nature of differential geometry. The result is one of the most readable and most valuable volumes in the series.

E. H. N.

An Introduction to the Mathematics of Map Projections. By R. K. Melluish. Pp. viii, 144. 8s. 6d. 1931. (Cambridge University Press)

This book will certainly fill a gap in English mathematical publications. It gives a comprehensive survey of the ends to be attained in the construction

of maps and of the various means which have been adopted for attaining them, with not merely general explanations but detailed formulae for working them out in practice.

All map-making is a compromise. A figure on a curved surface cannot be truly represented on a plane. If the shapes of infinitesimal portions are truly given, the scale will vary from point to point. If, on the other hand, areas throughout are accurate, there is necessarily distortion of shape. But these are two extremes. It requires the perusal of some such book as this to realise the innumerable compromises which have been used or proposed. These are duly set out, with explanations of their various advantages and disadvantages and their applicability to large or small portions of the earth's surface. Naturally, if the whole of the earth's surface is to be shown in one map, very special projections must be used and considerable distortion is inevitable. But even for large portions of the earth it is possible in various ways to secure *either* that areas are truly given *or* that the shapes of small elements are truly given. Not only are such possibilities mathematically developed here, but there are endless historical references to the map makers who have worked on different lines and to the mathematicians who have elaborated general principles.

In the detailed arrangement of topics there is some cross-division. They are grouped partly according to the geometrical bases of different classes of map construction, and partly according to the conditions which they fulfil. Thus Chapter II, entitled " The Simple Conic ", deals in the first instance with the idea of a map constructed on a cone cutting (or touching) the sphere (or spheroid) which represents the earth's surface, the cone being then developed into a plane; though the chapter includes very various methods directly or indirectly derived from this. Then Chapter V deals with Perspective Projections. On the other hand, Equal Area and Orthomorphic Projections are grouped together in Chapters III and IV respectively; so that there is necessarily some overlapping. This is perhaps of no great consequence, and it may be an advantage that (*e.g.*) in Chapter IV the somewhat difficult general discussion, based on Gauss, of the necessary conditions of orthomorphism is preceded by a reference to the special and important case of Mercator; though the very simple and interesting Stereographic Projection, which is referred to several times, hardly has sufficient prominence anywhere. Chapter VI includes miscellaneous projections which satisfy other conditions of some interest. Thus it is possible to construct a map so that the *directions* of all points from two specified points (which may be the positions of important places) are truly shown; or, again, so that the *distances* from two such points are correctly given. A more complicated condition, which involves the Calculus of Variations, is that the " Total Square Error " should be a minimum. This is defined quite early in the book as the sum of the squares of the errors of scale in the directions of meridian and parallel, summed for every point in the map; but its investigation is left for Chapter VI.

The primary result aimed at in each case is a formula for the position on the map of a point of given latitude and longitude. Presumably such results, generally approximations in the form of series, might be made the basis of arithmetical calculations, such as would be practically used in the construction of maps; but we are not carried beyond the (often very complicated) algebraical formulae. In the last three chapters various general questions are discussed, (*e.g.*) the nature of the distortion or deformation at different points in Chapter VII and the estimates of distance and direction of one point on a map from another in Chapter VIII. Finally, in Chapter IX, the best projection for a given country is discussed. It is obvious that this will vary not only according to the size of the country, but as it is more extended in latitude or in longitude. In an ingenious special method, devised by M. Tissot, the choice is made to depend on the form of the ellipse which will

surround the country mapped and fit its shape as nearly as possible. That the radii of such an ellipse inclined at angles of 45° to the principal axes should be a minimum (under specified conditions) secures that the scale error at the border of the map should be as small as possible, thus giving what may be considered the best result for the country in question.

Such in general outline is the scope of the book, but it must be understood that it is not easy reading. From beginning to end it bristles with formulae, many of them of interest rather to the map specialist than to the student who wishes to gain an idea of the general principles of map construction. For the former it provides a valuable storehouse of information and suggestion, such as he will hardly find elsewhere—at least within the same compass. And the rigour of the argument is agreeably lightened for the more general reader by historical and personal notes, explaining how and when the various methods were introduced or suggested.

The author must be congratulated on having brought together so much well-selected material within so moderate a compass. Unfortunately his powers of exposition in matters of detail are sometimes hardly equal to the task; and this is perhaps more marked in the simpler sections, where he is giving his own explanations, than in the more complicated investigations where he is, presumably, more closely following his authorities. It is not always clear whether the earth is being treated as a spheroid or a sphere; thus on p. 7 we are told that the radius of the earth is taken as unity, while in the same paragraph it is referred to as a spheroid. Just below we have the two very simple results, $\sin\theta_1 = nr_1$, $\sin\theta_2 = nr_2$; and it is said that, from these, n and the other constant which appears in the expression for the radius may be found, where the vague reference to " the other constant " is at this stage hopelessly unintelligible. There follows a reference to " another method of calculating the constants ", where it is not really *methods of calculation* which are in question, but radically distinct principles of map construction. At the beginning of Chapter III it is a little disconcerting to find it stressed in the first paragraph that, in a projection used by Ptolemy, *one* meridian only (the central one) and *three* parallels were divided truly and to learn two paragraphs later that *all* the parallels are divided truly (as we might have supposed). Such are a few early instances. " Anaximander of Milet " ! (p. 10) seems to betray a French source imperfectly anglicised. This is a trivial matter, but occasionally there are signs that the author has hardly grasped the essentials of the point which he is explaining. The statement attributed to M. Tissot (p. 98) that " any projection of one surface on to another is equivalent to an infinite number of orthogonal projections and variations from centres of similitude " may be a literal translation from the French, but is hardly intelligible in English, in the context in which it occurs. On p. 113, an explanation of the difference between rhumb-line and great circle (or geodesic) courses is followed by a reference to *three* different routes; it is added that " they are not always different, for the Mercator line is loxodromic and the gnomonic line a geodesic ", as though the distinction depended on the nature of the map projection.

It adds to the difficulty of criticism that, as the author explains in his preface, the student is left to fill in for himself " many details of the work which have, for the sake of brevity, been omitted from the text " ; but often either a more precise statement or a definite reference to what has gone before would make the reading far easier. It is irritating to find (p. 26), without any back reference : " If we choose a, as is usual, as in the simple conic with one standard parallel χ ", where we might be explicitly reminded that $a = \tan\chi - \chi$. Indeed, it is significant that not one of the endless equations in the book is numbered, so that, where the author deigns a back-reference (which is not often), he can only say " by the equations on p. 17 " and leave the reader to disentangle what he wants. But it is perhaps ungracious to

lay too much stress on minor weaknesses in what is on the whole a brave attempt to cope with the intricacies of a very difficult subject, the material for which has to be culled from the contributions of many workers throughout many ages.
P. J. H.

The Taylor Series. By P. DIENES. Pp. xii, 552. 30s. 1931. (Oxford University Press)

This book is bound to exercise a considerable influence on the study of advanced mathematics in our modern Universities, for nothing quite on the same lines has hitherto appeared in English. The book seems to have been written with a dual purpose; it falls into two parts of about equal length.

The first half of the book is occupied with the presentation of the elements of the theory of functions of a complex variable developed on the lines of Cauchy and Weierstrass. No doubt the author felt that the traditional lines of study of pure mathematics at the undergraduate stage are not well adapted as a preliminary to the study of modern developments of the subject. He therefore begins at the beginning and expounds "Real Variables", "Complex Algebra", "Infinite Series", "Elementary Functions". In a book of this kind it seems a little strange to treat the decimal representation of a number as fundamental, but this is only one example of several that could be quoted to show that this part of the book has been written with a didactic purpose, and that the author has his own views as to the way in which the student can best be helped to overcome the initial difficulties of this branch of study. Most of what is in this part of the book is already covered by existing English text-books and treatises, but no doubt many will find it a useful basis for the earlier part of an Honours course.

In the second and more valuable part of the book the purpose is to elucidate the problem of detecting the properties of a function from those of the coefficients in its Taylor expansion. Here the author has a clear field for, apart from Mandelbrojt's much smaller volume under the same title,* no other effort has been made to give a systematic account of this work in English. Here we have a clear account of the elements of the work of Hadamard on singularities, of Mittag-Leffler on representation in the star-domain, of Ostrowski on over-convergence. The concluding chapters are devoted to a study of the behaviour of a series on its circle of convergence and to the relation of divergence of the series to singularities of the function.

The subject is one in which great advances have been made in recent time and in which a good deal of work is being done at present. The book is a valuable introduction to a very interesting and live branch of Mathematics.
G. B. J.

Exercices de Calcul Différentiel et Intégral. By E. LAINÉ. Pp. iv, 146. 20 fr. 1931. (Vuibert)

This volume consists of solutions of 61 questions in differential and integral calculus set in degree examinations in Paris from 1920 to 1930. The range includes the theory of functions of a complex variable, ordinary and partial differential equations, and differential geometry, and the majority of the questions are of a searching and comprehensive nature. The order is chronological, for the very reason that, as the author remarks in the preface, almost every solution depends on more than one group of theorems and any classification must therefore be artificial. But a subject-index in which each problem is entered as often as seems appropriate adds to the utility of the collection.

The solutions are models for the candidate, not essays written round the questions. For general theorems that are being invoked M. Lainé naturally refers to the *Précis d'Analyse Mathématique* which he produced in 1927; the

* Rice Institute, 1927.

two volumes of this work failed to receive in the *Gazette* at the time the notice which they deserved, and an opportunity may be taken now to remedy the omission. The first volume deals with real variables and with analytic functions of the complex variable, the second with differential equations, ordinary and partial, and with differential geometry. In describing the work as a précis M. Lainé is not attempting to avoid responsibility; he hurries along where the formal developments present no difficulty, but he pays ample attention where principles have to be explained or where there is some delicacy in the analysis. For example, there is a discussion of singular solutions of an equation of the first order and the second degree that is far in advance of anything attempted in English books for the same class of students. It is the more surprising to find that the discussion of envelopes is naïve in the extreme, even de la Vallée Poussin's criticisms of the classical theory being ignored. The treatment of differential geometry is vectorial but not kinematical; the properties of geodesic torsion are allowed in consequence to seem almost accidental. Geodesic curvature is defined, but nothing is said of its relation to geodesics, and the Gauss-Bonnet theorem is not enunciated. On the other hand, this part of the second volume includes sections on contact transformations and on continuous groups which are warmly to be welcomed as enlarging the student's outlook. The section on partial differential equations was contributed by M. Bouligand, and is a characteristically lucid introduction to the subject.

It must be confessed that there are very few paragraphs of M. Lainé's *Précis* that might not have been written, and very few of the *Exercices* that might not have been set, a quarter of a century ago. To turn to Pólya and Szegö's *Aufgaben* from the one, or to the later chapters of Dienes' *Taylor Series* from the other, is to enter a new world. But the young mathematician can no more afford to be ignorant of the older world than to dispense with the multiplication table. M. Lainé's easy mastery is not to be acquired without a serious effort, and a study of his admirable solutions can be recommended, not only as a help for students who do not expect to proceed to more modern work, but also as excellent training for the most ambitious.

E. H. N.

Vector Analysis with Applications to Physics. By R. Gans. Translated by W. M. Deans. Pp. x, 164. 12s. 6d. 1932. (Blackie)

That a book should reach a sixth edition in its original language is evidence that it is in some ways well suited to the students for whom it is designed, and it is not surprising that Prof. Gans' well-known introduction to vector analysis has caught the enterprising eye of Messrs. Blackie & Son, to whom the English mathematician owes several important translations. Nevertheless, the book raises in an acute form the old question as to the standard of mathematical reasoning properly to be offered to the physicist. The various differential operations are defined by means of limits. For any of them it is then perfectly easy to deduce an expression in cartesian coordinates by means of an element in the shape of a rectilinear cell, to deduce an expression in curvilinear coordinates by means of an element in the shape of a curved cell, to associate a function with a surface integral by means of an element in the shape of a thin disc, and to investigate filar propositions by means of an element whose section is negligible in comparison with its length. In each individual case the process is quite proper: the mathematician can introduce a mean value theorem to discover what assumptions are implicit in the passage to the limit, and the physicist need not be discouraged from accepting the unanalysed arguments. But there is not the smallest reason, on the basis of these arguments, to suppose that the various processes are all dealing with the same function in the end. The assumption that because each definite process leads to one definite limit the limit is always the same is one which,

as the elementary example of $(x+y)/(x-y)$ should convince the most practical of physicists, is purely gratuitous, but it vitiates the whole of the theoretical work in this book.

As an outline of classical dynamics, hydrodynamics, and electrodynamics, from a discussion of the motion of a top to the solution of Lorentz's equations by means of retarded potentials, in a rather smaller compass than that of a single issue of the *Gazette*, the work is a remarkable achievement. The first mention of a vector that is changing relatively to a frame is almost unintelligible, and it is not everyone who finds the introduction of the Coriolis force a simplification of ideas, but in general the details are clear.

For the English edition, a considerable number of examples have been inserted, and solutions of these are given at the end of the book. The translation is sometimes stiff, but as a rule it is unobtrusive. There are, however, a few places where the English is nonsense and does sad injustice to the author. The printing maintains the excellent standard which the publishers have accustomed us to expect, but the buying of books cannot develop as a habit among mathematicians if prices must be fixed at the level imposed in this case. E. H. N.

1. **Leçons sur la Résistance des Fluides non visqueux. I.** By P. Painlevé. Pp. viii, 184. 40 fr. 1930. (Gauthier-Villars)

2. **Mécanique des Fluides.** By H. Villat. Pp. vii, 175. 50 fr. 1930. (Gauthier-Villars)

3. **L'Hydrodynamique et la Théorie Cinétique des Gaz.** By Y. Rocard. Pp. x, 160. 40 fr. 1932. (Gauthier-Villars)

M. Painlevé seeks to explain the paradox of d'Alembert, that a solid body moving with uniform velocity through a perfect fluid experiences no resistance to its motion. First, however, he will obtain general formulae for the forces and moments on a body moving in any manner whatever. Here, in this volume, he only develops the preliminaries for his theory; but he gives a summary of his results and some indications of what his theory will be. It appears that he will develop the theory of surfaces of discontinuity, and seek to explain their origin. If the velocity is anywhere discontinuous, particles originally in contact must separate. This separation takes place at the forward stagnation point, and is accompanied, according to M. Painlevé, by internal impulses. Indeed, M. Painlevé quotes a theorem of M. Hadamard, that in the absence of such impulses, surfaces of discontinuity could not develop in the fluid; and presumably he will look to these impulses to explain their origin. On this preliminary announcement, two remarks are to be made. First, according to modern notions, a fluid of zero viscosity must be regarded as the limit of a fluid of small viscosity; in such a fluid, vorticity has its origin in the boundary layer, and this remains true even when the viscosity becomes zero and the boundary layer infinitely thin. The result is then the production of a vortex sheet, or surface of discontinuity of velocity, in the fluid. Second, when the vorticity becomes zero, the Reynolds number becomes infinite, so that presumably the " physical " limit would be turbulent motion. Thus, however necessary it may be to stress the results that can arise from the fact that a real fluid is not a continuum in the strict mathematical sense, it is exceedingly doubtful if M. Painlevé will be able to give anything like an accurate picture of physical processes.

The book, apart from its ultimate object, which is serious and forbidding, is charming. The mathematics is so clear, and the whole so well written, that it is a joy to read. The chapters on harmonic functions should give pleasure to many mathematicians, and the treatment of elementary classical hydrodynamics will appeal to all teachers of the subject. M. Painlevé's faith in theory is a joy to the mathematical spirit. Whether it be true or false, it is

so comforting to read " On attribue communément tout l'honneur de la création des aéroplanes à l'empirisme et à l'audace des praticiens. C'est une complète erreur ". And when M. Painlevé says " L'influence de cette viscosité qui est faible si les mouvements du liquide sont réguliers et peu turbulents au sens de M. Boussinesc, devient considérable dès que l'agitation du liquide est intense, ou simplement notable ; les termes du premier degré qui interviennent dans la théorie de Poiseuille sont alors considérablement dépassés par ceux du second degré qui néglige cette théorie, et dont M. Boussinesc tient compte par une sorte de théorie statistique approchée "—when M. Painlevé says that, the reference to Stokes's theory as the theory of Poiseuille, the notions that the influence of viscosity is small, and the theory of Stokes applicable, till the motion becomes turbulent, the light-hearted treatment of turbulence, and the omission of any other name than that of M. Boussinesc, are all so delightfully naïve that only a churl could find fault.

2. M. Villat has written another book on hydrodynamics, mathematical throughout, with the mathematics well and clearly set out. The author himself states that he leaves comparisons with physical reality to others. The usual preliminaries of elementary hydrodynamics are set out very shortly, and M. Villat hurries on to a discussion of those recent theories that most readily lend themselves to elegant mathematical treatment. There are valuable chapters on harmonic functions and conformal representation; Prandtl's theory of the monoplane aerofoil is set out in a manner worthy of a mathematical treatise, and there is no attempt to over-simplify ; the method of obtaining the drag formula associated with the Karman vortex street is nearly correct—as correct as any I have seen ; and chapters on free streamline motion, including the extension to curved boundaries, and on Oseen's asymptotic theory, leading to solutions of mathematical elegance but doubtful physical validity, are included. The length of the discussion given of the Kutta-Joukowski lift formula for a solid with a salient point hardly seems justified in a book of this character, since the main results obtained are otherwise clear.

3. M. Rocard's book is timely, and provides a good introduction to the difficult subject of the application of the kinetic theory of gases to hydrodynamics. His methods and assumptions are always clear. Basing his extensions on Professor Chapman's work, M. Rocard has himself made valuable contributions to the subject in recent years. In this book, he opens with a readable account of the fundamentals of the kinetic theory of gases, and finds the usual equations of motion, and expressions for the viscosity for various molecular models. He deals with compressed and rarefied gases, and with mixtures of gases. The author's work on compressed gases has met with criticism, and he himself recognises, in an appendix, that his formula for the viscosity shows a decrease in viscosity with increasing pressure, which is contrary to experiment ; and he shows how other models would lead to a different result. It is in the last chapters, where the author considers boundary conditions, that his results will probably be treated with most reserve. He is unable to obtain anything like the usually accepted boundary conditions in a viscous fluid, and comes across paradoxes which suggest that near a wall the usual equations of hydrodynamics are inapplicable. A deeper investigation seems necessary. S. GOLDSTEIN.

Vorlesungen über einige Klassen nichtlinearer Integralgleichungen und Integro-differentialgleichungen nebst Anwendungen. By LEON LICHTENSTEIN. Pp. 162. Rm. 16.80. 1931. (Springer)

Lichtenstein's work on the problems of the propagation of surface waves of finite amplitude, on the figures of planets and on other similar problems is well known. In this book these various problems are handled as particular examples of a general method, and the author has simplified and unified them

in the process. Almost the first third of the book is devoted to a careful examination of the fundamental theory of non-linear integral equations, and this makes the book an essential part of a library on mathematical analysis.

Lichtenstein develops the theory by means of successive approximations, using a neat quadratic inequality for the attack. This inequality would evidently save much labour in many other branches of analysis. The book then takes up one after another various applications of the general theory to surface waves, to boundary problems, to elliptic differential equations in general and the dynamics of gravitating media, as well as hydrodynamics. Finally a chapter is devoted to the problem of particular non-linear integral equations of the type in which $\lambda\phi(s)$ is equated to a series of terms containing multiple integrals in which $\phi(s)$ occurs to various orders of multiplicity. A proof is given that under certain simple conditions there is always one value of λ for which there is a solution which is not identically zero.

The book as a whole does not concern itself with applied mathematics in the sense usual in England. It is not of the same family as Rayleigh's *Theory of Sound* or Lamb's *Hydrodynamics*, and it is evident that the author's interest is in the pure mathematics involved, in proving existence theorems and in obtaining methods of approximation which are logically rigorous. On the other hand, it must be admitted that in many problems of hydrodynamics, for example, the usual somewhat slap-dash methods are not particularly successful in explaining the facts as they actually are. Indeed, hydrodynamics, as a whole, is still in a state of chaos, and even if fruitful developments will most probably come from those most closely in touch with experiment there is considerable advantage in determining what pure theory really allows and what it does not permit. On this theoretical side the book makes a valuable contribution which is beyond dispute and can be recommended for what it achieves. P. J. D.

Einführung in die Wahrscheinlichkeitstheorie. By E. Kamke. Pp. vii, 182. Geh. RM. 10. Geb. RM. 11.50. 1932. (Hirzel)

Recent developments in mathematical physics, such as the formulation of the Principle of Indeterminacy, indicate that the ordinary laws of nature with their observed regularities refer essentially to the average effect produced by a very large number of particles, any one of which may behave in a way difficult or impossible to calculate. To deal with such average effects is the province of the theory of probability, which accordingly is becoming of ever-increasing importance. Unfortunately the foundations of this theory are unsound. The task of strengthening them is urgent, and this is the chief object of Dr. Kamke's book. In a historical note he points out the weaknesses of the old definition of probability, as the ratio of the number of favourable cases to the total number of *equally-likely* cases. As a general definition this is useless, for apart from a few special problems involving symmetry, such as those concerning coins, dice, or a pack of cards, there are no cases which can even plausibly be assumed as equally-likely. This objection applies particularly strongly to the applications of the greatest practical importance, such as insurance. Dr. Kamke raises a second objection to the old definition, namely that it does not, as sometimes supposed, lead to such a conclusion as that the relative frequency of throwing a three with a die in an infinite series of trials *must* tend to the limit one-sixth. Following von Mises, whose papers (from 1919) are asserted to be the first thoroughly logical treatment of probability, he defines a probability sequence as one in which the relative frequency tends to a limit, and defines the probability itself as this limit. Other definitions are necessary later on. It is claimed that upon this basis an abstract science can be built up, which is deducible from its definitions and axioms with the same accuracy as the science of geometry, and which, also like geometry, is in very good approximate correspondence with the results of experience.

The reader must judge for himself whether these far-reaching claims can be substantiated. Dr. Kamke's second objection to the old definition may be met by pointing out that the relative frequency does *not* tend to a limit, in the strict mathematical sense of that word. As he himself recognises, to postulate that the limit exists is to restrict the possibility of long runs in a way incompatible with the idea of independence. The theory built up on such a foundation, however logical in itself, appears to correspond with the results of experience less closely than could be desired. The book as a whole is so carefully written that it is surprising to find this difficulty inadequately discussed.

It is emphasised that in the author's treatment probability refers essentially to an infinite series of events, and not to a single event. This is quite contrary to the views of Keynes. It is possible that Kamke and Keynes may both be right; they may be discussing two different sciences, whose difference is disguised by the common use of the word probability. The limit definition of probability leads to the curious consequence that an event which occurs a finite number of times in an infinite series has the probability zero. According to the old definition, of course, an event which has the probability zero can never occur at all. There are several other points on which Dr. Kamke reaches conclusions opposed to those of older authors, including Poincaré.

The later portions of the book deal with the results of Bernoulli, Laplace, and Poisson, and with the Lexis dispersion theory. The economic situation has necessitated the shortening of the book, so several topics, such as geometrical probabilities and Bayes' theorem, have been reluctantly omitted. The last few pages contain a table of the probability integral, solutions (sometimes in full) to the eighteen examples in the text, a short bibliography and an index. H. T. H. Piaggio.

Yearbooks of the National Council of Teachers of Mathematics: (i) **The Teaching of Geometry.** Pp. x, 206. 1930. (ii) **Mathematics in Modern Life.** Pp. ix, 195. 1931. (iii) **The Teaching of Algebra.** Pp. ix, 180. 1932. $1.75 each. (Teacher's College, Columbia University, New York)

The National Council of Teachers of Mathematics plays in the United States much the same part as does the Mathematical Association in this country.

It was founded with the following objects: (i) to create and maintain interest in the teaching of mathematics; (ii) to keep the values of mathematics before the educational world; (iii) to help inexperienced teachers and to raise the general level of instruction in mathematics.

The official journal of the Society is called *The Mathematics Teacher*. In addition to this publication, the National Council has issued, one each year since 1926, volumes which are intended to serve the same purpose as the Reports of the Mathematical Association; the last three of these yearbooks are enumerated above.

There are, however, important differences of character:

(*a*) The yearbooks are more discursive and far more detailed in treatment than any M.A. Report; each forms a substantial volume containing from 180 to 200 pages.

(*b*) The contents are not compiled by a committee, but consist of a dozen or so articles on special subjects by well-known American teachers. Consequently there is more freedom of treatment and a franker expression of opinion on controversial subjects than is to be expected from a document drawn up by a representative body.

(*c*) It is also inevitable that in a series of articles by different authors there should be a certain amount of repetition, and, however judiciously the work of the general editor is performed, there must be points which escape discussion. Although the work of a committee is necessarily less lively and

provocative, it is constructed on a comprehensive plan which weighs in due proportion the various aspects of the subject and secures fair treatment for all.

(*d*) Reports of the Mathematical Association give invariably an authoritative account of the methods adopted by the most experienced teachers in this country, and, where there is any marked division of opinion, the attitude of each school of thought is described, and it is left to the reader to form his own verdict, although some indication may be given of which side appears to command the majority of support. For a reader unfamiliar with American methods of opinion, it is not easy to judge from these yearbooks how far the methods advocated by individual writers represent general practice. Many of them appear to be more in the nature of pioneer work than a summary of general teaching practice. This does not, of course, detract from their value; indeed in some respects it enhances it, if as seems likely the influence of these yearbooks is beginning to shape the course of American mathematical education.

It would be quite impracticable in the space here available to give any detailed account of the contents of these volumes. The object of this notice is to describe the principles on which they are constructed, so that those who are interested in the development of the subject in another country can avail themselves of this opportunity, which is provided by the National Council of Teachers.

There seems little doubt that much less time is given to the teaching of algebra and geometry in the United States than would be considered adequate here, and naturally the standard attained in school work is correspondingly lower. Further, changes which began to reform education here twenty-five years ago are only just beginning to modify on an extensive scale American methods. There is, however, much that British readers will find of interest in these yearbooks, and the series might with advantage be included in all training school libraries. C. V. D.

A School Calculus. By E. P. Oakes. Pp. xii, 232. 4s. 6d. 1931. (Pitmans)

A School Coordinate Geometry. By E. P. Oakes. Pp. xiv, 265. 4s. 6d. 1931. (Pitmans)

These two books cover the ground of coordinate geometry and calculus for the Additional Mathematics of School Certificate examinations and for the science group papers of Higher Certificates.

The *Calculus* calls for little comment. The exposition is sound on the whole, with one or two lapses; for instance, there is a tendency to "beg the converse", and in one place we are told that if $dy/dx=0$ and $d^2y/dx^2=0$, then "of course" the point can be neither a maximum nor a minimum. An interesting feature is the treatment of logarithmic and exponential functions; the essence of the method adopted is an informal proof that the limit for $h \to 0$ of $\log(1+h/x)/(h/x)$ is independent of x, and some skilful teaching will be needed to make the schoolboy appreciate the process. We may be grateful that the alternative treatment promised in the Preface has been omitted, since it can be found in any antiquated text-book.

The *Coordinate Geometry* seems less satisfactory, but this is not entirely the fault of the author. It may well be doubted whether coordinate geometry is a suitable subject for a school certificate examination; a certain amount of it must be done in connection with the calculus, but it should probably be limited to numerical work. If this is admitted, then the serious study of the subject can be taken up, at a later stage, by better methods.

It is true that new ideas should be introduced in terms of familiar ones, but is it therefore necessary to include the usual literal work connected with the circle? Cannot the ideas be acquired at the numerical stage? Then when the literal work is done there is no need for equations like $xx_1+yy_1=a^2$,

$yy_1 = 2a(x + x_1)$ and $xy_1 + yx_1 = 2c^2$ to be obtained separately, and, we may add, no need for $y = mx \pm a \sqrt{(1 + m^2)}$ to be obtained at all.

"Questions relating to polar coordinates instead of being scattered through the book have been presented together in a final chapter." Will this not ensure that in practice the student will make no use of these coordinates? Ought he not to have been using them at appropriate places throughout the course?

The bookwork about the factorization of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ is not very well stated.

The geometrical property on p. 191 ($SF = e \,.\, TE$, with the notation there adopted) is commonly attributed to J. C. Adams, not to Adam. There is a good historical introduction on pp. xi to xiv, and another in the Calculus volume.

It is inevitable that the conic sections should loom large in a text-book on coordinate geometry. Forty years ago the title of this book would have been "Conic Sections", and it is not clear that the contents justify the new name. Yet the book probably meets the needs of modern examinations very well. Are the examinations at fault? A. R.

College Algebra. By H. P. Pettitt and P. Luteyn. Pp. vi, 284. 12s. 1932. (Chapman & Hall)

This is a review of elementary algebra for students who have passed from a high school to an American University, and the aim is to stimulate interest by the introduction of new material. Some of the revision goes back to the stage of the (English) preparatory school, and on the other hand there are chapters on Determinants, Exponentials and Limits. The treatment of the new work is slight, probably because it is only intended to provide an outline of the subjects for students who, except in mathematical attainment, are fairly advanced.

It is difficult to see how the course would fit into the scheme of English teaching, and the price of the book is prohibitive. A. R.

A Geometry for Advanced Division, Central and Secondary Schools. Part II. By J. W. M. Gunn. Pp. vii, 116. With Answers. 2s. 1932. (Rivington)

Part I of this book was reviewed in these columns in July 1932. Its main purpose, as set out in the preface, is to stress the correlation of practical applications of Geometry with the logical training of Euclidean reasoning. Part II is of less distinction than the previous part. It consists mainly of formal work on Areas, Loci, and the Circle properties. This is all quite nicely, if conventionally, done. The propositions are well set out, and there are many well-grouped exercises. In one or two cases the author is inclined to rush his fences. Thus on the first page, he jumps to the conclusion that the area of a rectangle of sides l cm. and b cm. is lb sq. cm. by considering only the special case when l and b are integers. Following the formal work come four interesting pages on "Geometry Around Us", showing examples of geometrical design from architecture, weaving and pottery, with a page of ruler and compass designs for copy and enlargement. Finally comes a chapter of thirteen pages on elementary numerical Trigonometry.

The disappointment of this part is that the correlation between Plane and Solid Geometry, begun so promisingly in Part I, has been dropped. For instance, there was a chapter in the earlier part on Plan and Elevation, very good so far as it went, but manifestly incomplete. It was to be expected that there would be a chapter in the second part which would carry this forward to some useful end. It is suggested that such a chapter, together with a section on the circular sections of a sphere, would have been of more value than the very perfunctory last chapter on Trigonometry. Moreover, it is

unsatisfactory to attempt definitions and explanations of the Trigonometrical Ratios without some treatment, explicit if informal, of similarity.

There is an attractively written note on Elementary Surveying on pages 142 to 144. H. E. P.

Elementary Mechanics. By A. BUCKLEY and C. F. G. MACDERMOTT. Pp. viii, 216, xix. 4s. (Bell)

This book is essentially one for the use of a class with its teacher, and has unusual features designed to assist the teacher. All the bookwork in each chapter is put first, as concisely as possible, so that for the first reading at any rate it will rather serve as a guide to the teacher to be expanded and explained by him. Then follow three sets of examples, A, B, and C: A and B contain about seven corresponding questions each, such that when the class have had one set explained to them they may be expected to tackle the other: the C set contains considerably more examples, some quite easy. The chapters are all on the same plan and all quite short. In addition there are some 200 miscellaneous questions, providing altogether plenty of examples for practice. There is also an index.

The book is divided into two parts, Statics and Dynamics, but it would be quite possible to work them together. In the first part the introduction to the resolution of forces is not by the "Law of the Resolved Part", but by proving the Parallelogram of Forces; this latter is done by first proving the parallelogram law for vectors from the idea of displacements, and then showing that a force is a vector. In the chapter on Centre of Gravity it would have been better if the fact that a moment about a point is really a moment about a line had been made clear, since it is necessary to take moments about a line in the bookwork of the chapter. The Dynamics part opens with a welcome chapter on velocity and acceleration in which the acceleration is not assumed to be uniform; that is kept for the next chapter. The third chapter starts with the statement of Newton's Laws of Motion; from these $P=mf$ is obtained and absolute units are introduced at once. Hence in two respects the authors have not followed the recommendations of the Association's *Mechanics Report*. There are chapters on Relative Velocity and Projectiles, but circular motion and the coefficient of restitution are not included.

Pupils working on their own would not find this book suitable, since there are no worked-out examples, and the bookwork is very concise; for instance, it is left to the teacher to explain why the conservation of energy does not hold in the case of an impact. The book should, however, prove very useful to the teacher of a class starting Mechanics or doing it for the School Certificate, and working along the lines suggested. J. W. H.

The London Mathematical Society—Notes on the Preparation of Mathematical Papers. Pp. 20. 1s. 1932. (C. F. Hodgson & Son, Ltd.)

This pamphlet has been written with a double interest in view: first of all to authors of mathematical works containing mathematical formulae, and secondly to the printer, since, if the instructions and suggestions given are carried out, the work of the latter will be made easier. The authors, with the co-operation of their printers, have covered a very wide field, and they very clearly show that the setting up (the putting together of small pieces of metal by the compositor) of mathematical formulae is not by any means as straightforward and simple a matter as would appear from the printed result.

The authors have not only indicated a variety of settings which involve little trouble to the compositor, but they have also given suggestions which make for a better appearance in the arrangement of the printed mathematical formulae, at the same time giving alternative forms which are of equal value to the mathematician, but which make the work of the compositor easier.

The pamphlet should be in the hands of all such authors, as not only would

the final result, in so far as the appearance of the printing is concerned, be more satisfactory, but it would be found that if these suggestions were adopted the cost of the setting would be along more economical lines.

All those who combined to produce the work are to be congratulated on the thoroughness of their exposition of a difficult matter. JOHN M. JACK.

Valuation and Surplus. By R. K. LOCHHEAD. Pp. xiii, 99. 7s. 6d. 1932. (Cambridge University Press)

Students taking the earlier examinations of the Institute of Actuaries find that most of the syllabus is covered by approved text-books, and that consequently there is little need for them to read much outside these. For the final examinations, however, the position is different, and for these the student must obtain his knowledge of most of the examination syllabus from original papers scattered over such technical publications as the *Journal of the Institute of Actuaries.* Many students find some difficulty in obtaining a comprehensive view of the whole subject from their study of these papers, which treat, in full detail, particular and isolated sections of the subject. It is found, too, that some of these contributions, written for expert consideration, are not easily read by the student beginning his acquaintance with the subject.

To help students in this position the Institute of Actuaries Students' Society proposes to publish a series of books as guides to students in their reading, and the book under review is the first publication of the projected series. It gives a general introduction to the problems arising in connection with the valuation of the liabilities of a Life Assurance Company, and in the periodical distributions of surplus. It presents, in outline, a coherent description of the operations involved, referring the student at each stage to the appropriate papers in the *Journal* or elsewhere for the detailed study of each part of the subject. The book is not a text-book, as is emphasised by the President of the Institute in his Introduction, but it will greatly assist the student to attain the desired comprehensive view of the whole subject. Though many sections of the book would have been more useful to students if the author had ventured on a more complete treatment, the object of the book is ably achieved. The author does not merely furnish a commentary on the original papers; he supplies a fresh and lucid discussion of the difficulties which the student encounters in his reading.

The need for books of this character for the guidance of actuarial students has long been apparent, and the proposed new series, if it maintains the standard set by this first volume, will be warmly welcomed.

R. W. STURGEON.

A First and Second Course in Arithmetic; Examples only. By W. G. BORCHARDT. With Answers. Pp. viii, 282, lviii. 3s. 6d. 1932. (Rivingtons)

These examples, taken from the author's *First and Second Course in Arithmetic,* meet all the requirements of an ordinary Secondary School. At the beginning is a useful set of tables of weights and measures, at the end an equally useful set of tables of logarithms and other functions. T. M. A. C.

Outline of the History of Mathematics. By R. C. ARCHIBALD. Pp. 53. 30 cents. 1932. (Bulletin No. 18 of the Summer School for Engineering Teachers; The Society for the Promotion of Engineering Education.)

To achieve the writing of a history of mathematics up to the early years of the nineteenth century which should be informative, trustworthy and readable, in some fifty pages, might well be characterised, not only as improbable but as almost impossible. And if we had not been writing with Professor Archibald's pamphlet in front of us, we should have omitted the word "almost". Anything from the pen of Professor Archibald is bound to be informative and trustworthy; what amazes us is that he has been able to

effect so severe a condensation without leaving us a skeleton of dry bones. Instead he has given us a living organism, small in its dimensions but perfect in its proportions.

Of course, there are omissions. "Chinese and Japanese mathematics are not considered in such a skeleton survey, and the reference to mathematics of the Hindus is brief." But in tracing the main stream one cannot explore every tributary. Still, when overstepping his own bounds to mention the vector analysis of Gibbs and Heaviside, and Hamilton's quaternions, the author might have spared a word for Grassmann.

We are indebted to Professor Archibald for pointing out that on p. 34, l.12 "he" should read "Newton"; that on p. 45 it should be said that Ruffini was the first to demonstrate the impossibility of solving the general quintic in radicals; and that Neugebauer should have been mentioned for his remarkable new discoveries concerning Babylonian mathematics. And it is a pity that, following Cajori, the author has mistaken which part of Maclaurin's work it really was that Lagrange characterised as "a *chef d'œuvre* of geometry".

There is a useful list of 101 references to further literature; it is confined mainly to works in English. T. A. A. B.

The Queen of the Sciences. By E. T. Bell. Pp. 138. 5s. 6d. 1932. (The Williams and Wilkins Company, Baltimore; Baillière, Tindall & Cox, London)

This is one of the "Century of Progress" series which is to present "the essential features of those fundamental sciences which are the foundation stones of modern industry". Possibly a volume devoted to mathematics in such a series was expected to provide an answer to the sort of question we hear from the man in the 'bus: "Algebra! What's the good of *algebra*?" And, indeed, the book opens with three pages on the "Objects of Mathematics", in which Professor Bell talks of atoms and quanta. But then the author dismisses this as a "shoddy apology", and his attitude becomes that of Browning's song,

"Nay but you, who do not love her,
Is she not pure gold, my mistress?"

He is depicting the Queen of the Sciences, not Bacon's handmaiden of natural philosophy, and his warmest admiration is reserved for the logical beauties of mathematics, rather than for her triumphant applications.

So much is presented in so small a compass that it is hardly possible to indicate here the scope of the book. The aim is, of course, to describe as simply as possible the main lines of mathematical progress during the last hundred years or so. The sections which tell how algebra was reduced to a set of postulates, so that variation of the postulates gives rise to numerous algebras, are particularly good, though to refer to "the British algebraic school, Peacock, Gregory, Sir William Rowan Hamilton, Augustus De Morgan, and others", is surely unfair to Boole.

Most of us probably know those fits of depression in which we wonder if our teaching or research is really worth while; in such moods this book would prove an admirable tonic. We would also warmly recommend it to pupils just beginning to specialise; it will help them to broaden their mathematical understanding, and to develop a sense of the deep cultural significance of the science. Finally, we are indebted to Professor Bell for preserving that magnificent reply of Abel's when asked how he had been able to do so much in so short a time—"By studying the masters, not the pupils".

T. A. A. B.

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THE UNIVERSITY PRESS, GLASGOW

JOURNALS RECEIVED.

When no number is attached, no part has been received since a previous acknowledgment.

Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität. 8: 4.

Abstracts . . . from the Massachusetts Institute of Technology. 8.

American Journal of Mathematics. 53: 4.

American Mathematical Monthly. 38: 7, 8, 9.

Anales de la Sociedad Científica Argentina. 112: 1, 2, 3, 4.

Annales de la Société Polonaise de Mathématique.

Annals of Mathematics. 32: 4.

Anuario (Univ. Nac. de la Plata).

Berichte über die Verhandlungen der Akad. der Wiss. zu Leipzig: Math.-Phys. Klasse. 82: 3, 4, 5, 6; 83: 1.

Boletin Matematico. 4: 4, 5, 6, 7, 8, 9.

Boletin Matematico Elemental. 2: 8, 9, 10.

Boletín del Seminario Matemático Argentino.

Bollettino della Unione Matematica Italiana. 10: 4, 5.

Bulletin of the American Mathematical Society. 37: 7, 8, 9 i, 10, 11.

Bulletin of the Calcutta Mathematical Society. 20; 23: 2, 3.

Contribución al Estudio de las Ciencias Físicas y Matemáticas. 5: 2.

L'Enseignement Mathématique. 30: 1-2-3.

Half-Yearly Journal of the Mysore University.

Jahresbericht der Deutschen Mathematiker-Vereinigung. 41: 1-4.

Japanese Journal of Mathematics. 8: 2.

Journal of the Faculty of Science, Hokkaido. Ser. 1. 1: 1, 2.

Journal of the Indian Mathematical Society. 19: 3, 4, 5.

Journal of the London Mathematical Society. 6: 2, 3, 4.

Journal of the Mathematical Association of Japan. 13: 4-5.

Journal de la Société Physico-Mathématique de Léningrade.

Mathematical Notes.

Mathematics Teacher. 23: 8.

Memoria (Univ. Nac. de la Plata).

Monatshefte für Mathematik und Physik.

Nieuw Archief voor Wiskunde.

Periodico di Matematiche. Ser. 4. 11: 5.

Proceedings of the Edinburgh Mathematical Society.

Proceedings of the Physico-Mathematical Society of Japan. Ser. 3. 13: 7, 8, 9, 10.

Publicaciones de la Facultad de Ciencias Físico-Matemáticas Universidad Nacional de la Plata.

Publications de la Faculté des Sciences de Masaryk. 138, 140, 143, 144.

Revista de Ciencias (Peru). 389-390, 391-393.

Revista Matemática Hispano-Americana (Madrid).

Revue Semestrielle des Publications Mathématiques.

School Science and Mathematics. 31: 7, 8, 9.

Sitzungsberichte der Berliner Mathematischen Gesellschaft.

Studia Mathematica.

Unterrichtsblätter für Mathematik und Naturwissenschaften. 37: 9, 10, 11, 12.

Wiskundige Opgaven met de Oplossingen.

THE LIBRARY.

160 Castle Hill, Reading.

The Librarian reports gifts as follows:

From Mr. **T. A. A. Broadbent,** a text book by H. H. Dalaker and H. E. Hartig; and

W. A. Granville Differential and Integral Calculus - - - - - 1904

From the Author:

F. G. W. Brown Plane Trigonometry - - - - - - - 1930
Being Part II of *Progressive Trigonometry*.

From Mr. **A. S. Gosset-Tanner,** text-books by W. H. Besant (2), W. H. Drew, W. Garnett, H. Godfray, H. S. Jones, J. Lightfoot, C. Pendlebury, S. Parkinson, J. H. Smith, H. A. Stern and W. H. Topham, I. Todhunter (6), J. Wood, A. Wrigley, with

C. Leadbetter New Tables of the Planets, the Fixed Stars, and the First Satellite of Jupiter (vol. 2) - - - - - 1742

H. A. Morgan Problems and Examples - - - - - - 1858
From Jesus College Papers. *This copy has the ingenious autograph of C. L. Dodgson.*

From Miss **H. P. Hudson:**

J. L. Coolidge Algebraic Plane Curves - - - - - - - 1931

From Dr. **G. J. Lidstone:**

R. Todhunter Compound Interest and Annuities-Certain (3: R. C. Simmonds and T. P. Thompson) - - - - - - 1931

From Prof. **E. H. Neville:**

Ramchundra Maxima and Minima - - - - - - - 1850
The original Calcutta edition of the work of which De Morgan thought so highly that he persuaded the East India Company to finance a reprint, as nearly as possible in facsimile, which appeared in 1859. A copy of this later edition was in Mr. Greenstreet's bequest.

J. M. F. Wright Conic Sections and other Curves - - - - - 1836

From Mr. **B. A. Swinden,** vols. 50-53 of the *Messenger of Mathematics.* By gifts and exchanges the run left by Mr. Greenstreet was completed precisely as far as vol. 49, so only 5 volumes are wanted to complete the set to the end.

From Mr. **F. P. White:**

G. Loria Storia delle Matematiche; II: I Secoli xvi e xvii - 1931
Vol. I was given to the Library by Mr. Greenstreet.

From Miss R. A. Clayton, Mr. L. S. Milward, Prof. R. H. Pinkerton, and Miss G. M. Weighall, collections of back numbers of the *Gazette*.

From the Massachusetts Institute of Technology, missing numbers from their series of *Abstracts*, rendering the Library set complete.

The following have been bought:

F. Cajori Teaching and History of Mathematics in the United States - - - - - - - - - - 1890
A report published by the Bureau of Education. Includes details of many short-lived mathematical journals published in the States.

J. S. Ersch — Literatur der Mathematik, Natur- und Gewerbs-Kunde seit der Mitte des 18 Jahrhunderts {: F. W. Schweigger-Seidel} - - - - - - 1828
A useful work of reference in a mathematical library.

C. J. Hargreave — An Essay on the Resolution of Algebraic Equations 1866
Posthumous, with a preface by G. Salmon.

J. Milnes — Sectionum Conicarum Elementa {3} - - - - 1723
Enlarged from the previous edition, 1712, which was given by Mr. Milne.

J. Playfair — Elements of Geometry {8: W. Wallace}. - - - - 1831
Presentation copy from the Editor to G. B. Airy; the leaves have not been cut but the original binding has been stripped!

Duke of Somerset
Alternate Circles {1 (1850) rep.} - - - - - - 1851
The first edition seems to have consisted only of presentation copies. The substitution of a publisher's name for a printer's and the change in date, are the only alterations, even the errata noted in the original issue not being corrected in the text. But the binding is cloth, not ornamented calf.

W. Wallace — Geometrical Theorems and Analytical Formulae - 1839
With applications to geodetical problems. An unattractive ill-arranged collection, containing, among other matter, an account of isogonal conjugates.

Mathematical Questions and Solutions, New Series, 5-6 - - - - 1918
Completing the set bequeathed by Mr. Greenstreet.

Unterrichtsblätter für Mathematik und Naturwissenschaften.

There is no equivalent of the Mathematical Association in Germany, for a single Society includes in its scope mathematics and the exact and biological sciences. This Society has exchanged journals with the M.A. for some years past, and the recent purchase of a run of the early volumes gives the Library a set complete from the beginning in 1895.

Calcutta Mathematical Society.

To celebrate its twentieth year, this Society decided in 1928 to make of the current volume of its *Bulletin*, a commemoration volume, but the production was necessarily slow, and meanwhile there has been a gap in the series, ordinary volumes having appeared at natural intervals. The gap is now filled by an interesting volume which contains 27 papers from all parts of the world. The English contributors are Larmor, Lamb, Dyson, Forsyth, Whittaker, and Hardy and Littlewood.

LONDON BRANCH.

REPORT OF MEETINGS.

At the invitation of the Chairman, Mr. A. W. Siddons, the summer meeting was held at Harrow School on Saturday, 6th June, 1931. There was a good attendance of 67 members and friends of members.

After viewing the boys' quarters at "Rendalls", visitors were conducted through the Art Schools, the Speech Room and War Memorial Buildings, the Vaughan Library and the Chapel. Tea in the School tuck shop followed. Groups were then formed to visit the playing fields, the workshops and "Ducker". The visit was favoured by perfect weather. The thanks of the Branch are due to Mr. Siddons, and to his colleagues, Messrs. Hughes, Snell and Calvert, who assisted their visitors to view the School.

At the opening meeting of the autumn session there was a record attendance of 120 to hear a challenging paper by Mr. B. C. Wallis, the Chief Examiner to

the London County Council, on " Arithmetic from the Examiner's Point of View ". Mr. Wallis stated as his main proposition that *there has been no definite improvement in the quality of the arithmetic produced in examinations during the last twenty-five years.* He asserted that no child should produce for inspection any result which he has not tried to check, and which he does not feel sure is correct. Current arithmetic was greatly overloaded, and was dominated by a rule-of-thumb attitude—hence formal neatness requiring sums set out in many words and to a fixed pattern. Ideal arithmetic should make little use of words. An intuitive solution to a problem or sum should be accepted. Facility in deductive demonstration was not to be expected from a child of mental age less than 16. Two cardinal arithmetical sins are " dodging " inverse proportion and the use of an unknown x.

In conclusion Mr. Wallis suggested some constructive ideas. When the four rules are known, the notion of *sequences* may be developed so as to cover most of arithmetic. The time thus saved may be used to develop arithmetical thinking. The ensuing discussion was appropriately vigorous.

A meeting was held at Bedford College on Saturday, 7th November, when the Chairman, Mr. A. W. Siddons, gave a paper on " The First Two Years of Geometry in a Secondary or Preparatory School ". There was an excellent attendance of over 120 members and visitors.

Mr. Siddons suggested that in the first year, scale-drawing should be the chief aim. In the first term there should be handling of models and making of plans. Some theorems—particularly those depending on the angle properties of a triangle—should be studied in the second term. The third term should bring in work with compasses, leading to the idea of congruence. During the second year the work could be taken in three sections—the " parallelogram and mid-point " group, the " area and Pythagoras " group, and the " circle " group. The order in which these were taken was immaterial. Work on loci, particularly exercises in three dimensions, could also be done.

Mr. Siddons concluded with a hearty condemnation of the use of a fixed sequence in the teaching of geometry.

The Presidential Address was given on Saturday, 5th December, at Bedford College by Mr. J. W. N. Sullivan on " Mathematics and Culture ". There was an attendance of 62 members and 5 visitors.

Mr. Sullivan suggested that culture implied a combination of information and sensibility. The refinement of sensibility was the aim of culture. Mathematicians sometimes appeared persons of little culture, possibly because mathematics was a curiously isolated activity. Originally it was supposed to give knowledge of ultimate truth and of necessary truths. We now realised that mathematics gave us no knowledge of any world that was independent of the human mind. Although the external universe appeared to obey mathematical laws, yet it could be maintained that mathematics and logic were but devices for adapting ourselves to the world.

We could gain an insight into the cultural value of mathematics by enquiring what mathematics did for us. Was it a science, an art or a game ? Its history suggested an art. Notably does it compare with music in that both activities develop of themselves. If it was an art it should give aesthetic pleasure—and there was much evidence of this.

Mathematics refined our intellectual capacities ; such notions as the general concept of number were really an addition to the resources of the mind. As mathematics came to play an increasing part in our study of the universe so would knowledge of it be an increasing element of culture.

C. T. Daltry, Hon. Sec.

ERRATUM.

Vol. XV, p. 487, l. 7 up. *For* portion *read* positive.

QUEENSLAND BRANCH.

Report for the Year 1930-31.

The 1930 Annual Meeting was held at the University on March 28th. The Annual Report and Balance Sheet were presented to the meeting and were adopted, after which officers for the year were elected. Professor Priestley in his presidential address spoke on "An Old Algebra".

During the year three General Meetings were held: the first at the Boys' Grammar School on May 23rd. At this meeting Mr. S. Stephenson read a paper on "Mathematical Freaks". At the second, held on August 8th, at the University, Dr. E. F. Simonds read a paper on "The Early History of the Cubic", and at the third held at the University on October 31st, Mr. A. J. Stoney read a paper on "The Use of Mathematics in Engineering".

The number of members of the branch is 34, of whom 11 are full members of the Mathematical Association; this shows an increase on last year's figures. Copies of the *Gazette* are received regularly and are circulated among associate members. The statement of receipts and expenses reveals a satisfactory state of affairs, the balance in hand being £6 10s. 11d., which is practically the same as at the end of last year.

The attendance at meetings has been satisfactory, and members continue to take an interest in the affairs of the branch.

Officers: *President*: Professor H. J. Priestley. *Vice-Presidents*: Mr. S. Stephenson, Dr. E. F. Simonds. *Hon. Secretary and Treasurer*: Mr. J. P. McCarthy. *Committee*: Miss E. H. Raybould, Miss E. M. B. Cribb, Mr. J. Waddle, Mr. E. W. Jones, Mr. R. A. Kerr.

MANCHESTER BRANCH.

Two meetings have been held during the Lent term. On January 19th, Mr. F. Bowman of the Manchester College of Technology gave a paper on "Beginnings in the Calculus", and treating the subject from its earliest stages, he outlined a course which would adequately prepare pupils for more advanced work at a college or university. On February 24th, a joint meeting with the University Mathematical Society was held at the University. Mr. H. L. Joseland gave a highly instructive paper entitled "Some Mathematical Reminiscences". The various aspects of mathematical teaching and examination as seen by pupil, master and inspector were brought before us in their sequence of historical controversies. A. I. G.

VICTORIA BRANCH.

Report for the Year 1930.

The office-bearers for the year 1930 were:

Hon. President: Professor Nanson. *President*: Professor Cherry. *Vice-Presidents*: Dr. J. M. Baldwin, Miss K. Gilman Jones, Professor Michell, Mr. D. K. Picken, Mr. M. S. Sharman. *Committee*: Mr. C. B. Lane, Miss J. T. Flynn, Miss W. Waddell. *Treasurer*: Mr. F. W. Campbell. *Secretaries*: Mr. R. J. A. Barnard, Mr. J. L. Griffiths.

There were in all 28 members of the branch in 1930, of whom 11 were members of the parent society.

Five meetings were held in the year at all of which there were good attendances.

In April, Professor Cherry spoke on methods of dealing with the Calculus required for the Honours Course for the School Leaving Examinations. He dealt especially with the fundamental limits—the derivative and the integral—and the introduction of the exponential and logarithmic functions.

In June, Professor Michell gave an address on the "Arithmetical Definition of the Trigonometrical Functions". He explained how the complete analytical determination of the properties of the sine and cosine for all rational and irrational values of the argument could be obtained if we started from the definitions.

$$\cos(x+y)=\cos x\cos y-\sin x\sin y,$$
$$\sin(x+y)=\sin x\cos y+\cos x\sin y,$$
$$\sin^2 x+\cos^2 x=1.$$

In July, Mr. S. L. Hughes dealt with the teaching of Arithmetic and Elementary Algebra.

In September, Mr. Seitz compared the syllabuses of the Leaving Examinations in the different states of the Commonwealth and explained the principles underlying them.

In October, Mr. Gundersen explained Gauss' work on regular polygons and in particular showed how the construction of the seventeen-sided regular polygon was reducible to the graphical solution of a number of quadratic equations.

R. J. A. Barnard, Hon. Sec.

STUDIA MATHEMATICA.

This Polish annual, of which the third volume has just appeared, is edited by Professors Steinhaus and Banach. They have endeavoured to attract papers on the theory of functions of a real variable, and in particular on the theory of Fourier series and related topics; in its short career the journal has contained many articles of a high standard of excellence in this abstruse but fascinating field.

We trust that the rumour that no more volumes are to appear is entirely without foundation. Doubtless the Polish Government will realise the important part that *Studia Mathematica* is playing in indicating to the outside world the virility of scientific research in Poland.

BOOKS RECEIVED FOR REVIEW.

H. Abson. *First Trigonometry for Schools.* Pp. ix, 288. 3s. 6d. 1931. (University Tutorial Press)

H. Bateman. *Partial Differential Equations of Mathematical Physics.* Pp. xxii, 522. 42s. 1932. (Cambridge)

T. C. Batten and M. W. Brown. *A School Certificate Algebra.* Pp. ix, 460. 5s. With Answers, 5s. 6d. 1931. (John Murray)

T. C. Batten and M. W. Brown. *A School Algebra. Part II.* Pp. viii, 199-521. 4s. 6d. With Answers, 5s. 1931. (John Murray)

W. S. Beard. *Everyday Arithmetic and Accounts: Second Year Course.* Pp. 70, xviii. Paper, 1s.: cloth boards, 1s. 6d. 1931. (Oxford University Press)

L. Bieberbach. *Projektive Geometrie.* Pp. vi, 190. Rm. 7.80. 1931. (Teubner).

Mary E. Boole. *Collected Works.* Vols. I-IV. Edited by E. M. Cobham. Pp. xli, 1566. 60s. 1931. (Daniel, London)

F. Bowman. *Elementary Calculus.* Pp. vii, 286. 6s. 6d. 1931. (Longmans, Green)

H. Brandenburg. *Siebenstellige trigonometrische Tafel.* Second edition. Pp. xxviii, 340. Rm. 36. 1931. (Alfred Lorentz, Leipzig)

D. Brunt. *The Combination of Observations.* Second edition. Pp. x, 240. 12s. 6d. 1931. (Cambridge)

E. G. Coker and L. N. G. Filon. *A Treatise on Photo-Elasticity.* Pp. xviii, 720. 50s. 1931. (Cambridge)

J. L. Coolidge. *A Treatise on Algebraic Plane Curves.* Pp. xxiv, 513. 30s. 1931. (Oxford ; at the Clarendon Press)

B. H. Crenshaw, Z. M. Pirenian and **T. M. Simpson.** *Mathematics of Finance, preceded by Elementary Commercial Algebra.* Pp. xiii, 383. 15s. 1930. (Pitman)

L. Crosland. *First Ideas in the Calculus.* Pp. xi, 132. 2s. 6d. [1931.] (Ginn)

L. Crosland. *School Mathematics for first year students in secondary and other schools.* With Answers. Pp. x, 308. 3s. 6d. 1932. (Macmillan)

C. G. Darwin. *The New Conceptions of Matter.* Pp. viii, 192. 10s. 6d. 1931. (Bell)

H. T. Davis. *Philosophy and Modern Science.* Pp. xiv, 335. $3.50. 1931. (Principia Press, Bloomington, Indiana)

P. Dienes. *The Taylor Series : an Introduction to the Theory of Functions of a Complex Variable.* Pp. xii, 552. 30s. 1931. (Oxford ; at the Clarendon Press)

J. Dollon. *Problèmes d'Agrégation. Mathématiques Élémentaires.*) Pp. 92. 15 fr. 1931. (Librairie Vuibert)

J. Dougall. *Four-Figure Mathematical Tables.* Pp. 32. 1s. 1931. (Blackie)

M. J. van Driel. *Oneven Toovervierkanten Zijn Geen Puzzles.* Pp. 181, 97. No price. 1931. (Versluys, Amsterdam)

C. V. Durell. *The Teaching of Elementary Algebra.* Pp. viii, 136. 3s. 6d. 1931. (Bell)

H. G. Forder. *Higher Course Geometry.* Pp. x, 264. 6s. 1931. (Camb. Univ. Press)

G. Gamow. *Constitution of Atomic Nuclei and Radioactivity.* Pp. viii, 114. 10s. 6d. 1931. (Oxford)

R. Gans. *Vector Analysis with Applications to Physics.* Pp. ix, 163. 12s. 6d. 1932. (Blackie)

G. Garcia. *Teorema Generalizado de la Aceleracion en el Movimento Compuesto.* Pp. 8. 1931. (Chorrillos, Imp. de la Escuela Militar)

G. Garcia. *Ecuaciones Universales Completas de la Dinámica.* Pp. 25. 1931. (Imp. Peruana)

G. Garcia. *Teoremas y Fórmula Fundamental de la Resistencia de Materiales.* Pp. 10. 1931. (Chorrillos ; Imp. de la Escuela Militar)

G. Garcia. *La Reforma de la Mecánica Celeste de Hoëne Wronsky.* Pp. 40. 1931. (Lima)

C. Godfrey and **A. W. Siddons.** *The Teaching of Elementary Mathematics.* Pp. xi, 322. 6s. 6d. 1931. (Camb. Univ. Press)

J. W. M. Gunn. *A Geometry for Advanced Division, Central and Secondary Schools.* Part I. With Answers. Pp. viii, 104, iv. 2s. 1931. (Rivingtons)

E. W. Hobson. *The Theory of Spherical and Ellipsoidal Harmonics.* Pp. xi, 500. 37s. 6d. 1931. (Camb. Univ. Press)

A. H. Jameson. *Contour Geometry.* Pp. vii, 158. 7s. 6d. 1932. (Pitman)

H. Jeffreys. *Cartesian Tensors.* Pp. vii, 93. 5s. 1931. (Camb. Univ. Press)

H. Jeffreys. *Operational Methods in Mathematical Physics.* 2nd Ed. Pp. viii, 117. 6s. 6d. 1931. (Camb. Univ. Press)

C. Jennings and **R. L. W. Tobutt.** *Army Mathematics.* Second edition. Part I. *For Army Second Class Certificate.* Pp. 64. Part II. *For Army First Class Certificate.* Pp. vii, 248. Bound in one. 6s. 6d. 1932. (Oxford University Press)

E. Lainé. *Exercises de Calcul Différentiel et Intégral.* Pp. 146. 20 fr. 1931. (Librairie Vuibert)

F. A. Lindemann. *The Physical Significance of the Quantum Theory.* Pp. vi, 148. 7s. 6d. 1932. (Oxford; at the Clarendon Press)

B. B. Low. *Mathematics : A Text-Book for Technical Students.* Pp. vii, 448. 12s. 6d. 1931. (Longmans)

R. K. Melluish. *An Introduction to the Mathematics of Map Projection.* Pp. viii. 145. 8s. 6d. 1931. (Camb. Univ. Press)

E. P. Oakes. *A School Calculus.* Pp. xii, 232. 4s. 6d. 1931. (Pitman)

A. S. Pratt and **E. E. Kitchener.** *Junior Primary Tests in Arithmetic.* Pp. 128. Without Answers, 1s. With Answers, 1s. 6d. 1932. (Harrap)

J. Prescott and **H. V. Lowry.** *Elementary Trigonometry.* Pp. xi, 444. 5s. 1932. (Longmans)

Y. Rocard. *L'Hydrodynamique et la Théorie Cinétique des Gaz.* Pp. x, 160. 40 fr. 1932. (Gauthier-Villars)

F. Schilling. *Die Pseudosphäre und die nichteuklidische Geometrie.* Pp. vi, 70. Geh. Rm. 3. Geb. Rm. 4. 1931. (Teubner)

P. F. Smith and **W. R. Longley.** *Intermediate Calculus.* Pp. xiii, 457. 16s. 6d. [1931.] (Ginn)

W. Smith. *The New Graded Arithmetics.* I, II. Paper, 5d. Cloth, 7d. III. Paper, 6d. Cloth, 8d. IV. Paper, 7d. Cloth, 9d. V. Paper, 8d. Cloth, 10d. *Teacher's Books.* I-III. 1s. 6d. IV, V. 2s. 1924-1931. (Oxford Univ. Press)

W. Smith. *Test Papers in Arithmetic.* Part I. Pp. 48. 9d. Part II. Pp. 42. 9d. 1932. (John Murray)

R. Smyth. *The Quadrilateral. An investigation of its chief properties and a synopsis of old theorems and problems.* Pp. 42. No price stated. [1931.] ("Home Words" Publishing Co., London)

L. H. C. Tippett. *The Methods of Statistics.* Pp. 222. 15s. 1931. (Williams & Norgate)

J. F. Tocher. *What is "Probable Error"?* Pp. 64. (Institute of Chemistry)

L. Toft and **A. D. D. McKay.** *Practical Mathematics.* Pp. v, 594. 16s. 1931. (Pitman)

C. O. Tuckey and **P. W. C. Hollowell.** *A School Geometry.* Pp. xvii, 340. 4s. 6d. 1931. (Christophers)

B. L. van der Waerden. *Die gruppentheoretische Methode in der Quantenmechanik.* Pp. viii, 158. Geh. Rm. 9. Geb. Rm. 9.90. 1932. (Springer)

James Clerk Maxwell: a Commemoration Volume, 1831-1931. Pp. 146. 6s. 1931. (Camb. Univ. Press)

Mathematical Tables. Vol. I. Circular and Hyperbolic Functions: Exponential Sine and Cosine Integrals: Factorial (Gamma) and Derived Functions: Integrals of Probability Integral. Prepared by the Committee of the British Association for the Calculation of Mathematical Tables. Pp. xxxvi, 72. 10s. 1931. (London, Office of the British Association, Burlington House)

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., Derby School, Derby, to whom all inquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the text-books on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. The names of those sending the questions will not be published.

D. pages 9 - 12

THE LIBRARY.

160 Castle Hill, Reading.

The Librarian reports gifts as follows:

From Mr. **F. C. Boon**:

M. J. van Driel Oneven Tooververkanten zijn Geen Puzzles - - 1931

From Sir **A. S. Eddington**:

F. A. Lindemann Physical Significance of Quantum Theory - - 1932

From Mr. **W. Hope-Jones**:

E. A. G. Lamborn Reason in Arithmetic - - - - - - - 1930

From Mr. **E. M. Langley**:

M. Cantor Geschichte der Mathematik. I, II {2 (1900) rep.} 1880, 1913

M. Chasles Aperçu Historique {2, *i.e.* 1 (1837) rep.} - - - - 1875

W. Emerson Algebra - - - - - - - - - - - 1764
As in others of his books, the author's name is not on the title-page, but there is a signed Preface.

J. Landen Mathematical Lucubrations - - - - - - 1755
" Containing new Improvements in various Branches of the Mathematics." Includes the geometrical basis of the transformation in elliptic functions known by the author's name.

I. Newton Philosophiæ Naturalis Principia Mathematica
{T. Le Seur et F. Jacquier; J. M. F. Wright} (2 vols.) - 1833
A line-for-line reprint in two volumes of the edition of 1822, which was in four. It is the " Jesuits' Edition " corrected. *Vols. 2-4 of the earlier edition are in the Library. Has any member Vol. 1 to give?*

W. Thomson and P. G. Tait
Treatise on Natural Philosophy; I {2} (2 vols.) 1886, 1883

From Mr. **B. J. Lewsley**, *Nouvelles Annales de Mathématiques*, Sér. 6, Vol. 1 (1926).

From Rev. **J. J. Milne**:

B. Bridge Plane Trigonometry {3} - - - - - - 1822
" With the Method of constructing Trigonometrical Tables."

E. Catalan Théorèmes et Problèmes de Géométrie Elémentaire
{6} - - - - - - - - - - - - - 1879
Completely reset, with the folding copper-plates of the previous editions replaced by white-on-black text-figures.

C. J. Myers Differential Calculus - - - - - - - 1827

From the compiler:

A. S. Pratt Higher Certificate Applied Mathematics Test
Papers {2} - - - - - - - - - - 1931
Higher Certificate Mathematical Test Papers {3} - 1930
Junior Test Examinations in Mathematics - - 1930
Matriculation Advanced Mathematics Test Papers - 1929
Test Examinations in Mathematics {3} - - - 1928
Test Examinations in Mechanics - - - - 1929

From the authors:

A. S. Pratt and E. E. Kitchener
Junior Mathematics - - - - - - - 1931

From Mr. R. W. Sturgeon :

R. K. LOCHHEAD	Valuation and Surplus	1932

From Mr. F. Underwood :

A. MALCOLM	A New System of Arithmetick	1730
	"Theorical and Practical. Wherein the Science of Numbers is demonstrated in a Regular Course from its First Principles, thro' all the Parts and Branches thereof; either known to the Ancients, or owing to the Improvements of the Moderns. The Practice and Application to the Affairs of Life and Commerce being also fully explained: so as to make the Whole a Complete System of Theory, for the Purposes of Men of Science; and of Practice, for Men of Business."	

From Rev. S. H. Clarke and Mr. C. E. Williams, collections of back numbers of the *Gazette.*

The following have been bought:

M. GHETALDUS	Nonnullæ Propositiones de Parabola	1603
	Promotus Archimedis	1603
T. GIBSON	Syntaxis Mathematica	1655
	"Or, a Construction of the harder Problems of Geometry: with so much of the Conicks as is therefore requisite . . ."	
F. À. SCHOOTEN	De Organica Conicarum Sectionum Descriptione	1646
	Principia Matheseos Universalis	1651
F. W. WESTAWAY	Craftsmanship in the Teaching of Elementary Mathematics	1931

INTERNATIONAL ASSOCIATION FOR PROMOTING THE STUDY OF QUATERNIONS AND ALLIED SYSTEMS OF MATHEMATICS.

By the kindness of Prof. J. B. Shaw, the set of Reports and Bulletins of this Association, which was formed in 1900 and came to an end in 1913 for lack of support, is now complete.

THE MESSENGER OF MATHEMATICS.

By purchase of the few volumes required, the set of 58 volumes, 1872-1929 has been completed. This valuable training-ground for mathematical writers did not survive the editor, J. W. L. Glaisher, an appreciation of whom by G. H. Hardy is in the last volume. There are General Indexes to vols. 1-25 and vols. 26-58.

Of the 58 volumes, 42 came to the Association from Mr. Greenstreet. The *Messenger* had a predecessor, *The Oxford, Cambridge and Dublin Messenger of Mathematics,* of which five volumes were published; the Library has the first three of these.

NOUVELLES ANNALES DE MATHÉMATIQUES.

A victim, like *L'Intermédiaire des Mathématiciens,* of war and post-war conditions. The fourth series ended normally in 1920, the fifth consisted of three volumes, 1922-1924, and the sixth of two only, 1925-27. By gifts, exchanges, and purchases the set is now complete in 84 volumes, 1842-1927. The Library has *L'Intermédiaire* complete in 31 volumes, 1894-1925.

BOOKS RECEIVED FOR REVIEW.

R. C. Archibald. *Outline of the History of Mathematics.* Pp. 54. 30 cents. 1932. (Bulletin No. 18 of the Summer School for Engineering Teachers; the Society for the Promotion of Engineering Education)

E. T. Bell. *The Queen of the Sciences.* Pp. 138. 5s. 6d. 1931. (Williams and Wilkins Company, Baltimore; Baillière, Tindall and Cox, London)

A. S. Besicovitch. *Almost Periodic Functions.* Pp. xiii, 180. 12s. 6d. 1932. (Cambridge)

L. Bieberbach. *Analytische Geometrie.* 2 Auflage. Pp. iv, 142. RM. 5.95. 1932. (Teubner)

W. G. Borchardt. *A First and Second Course in Arithmetic. Examples only.* With answers. Pp. viii, 282, lviii. 3s. 6d. 1932. (Rivingtons)

E. W. Brown. *Elements of the Theory of Resonance illustrated by the Motion of a Pendulum.* Pp. 60. 3s. 6d. 1932. (Cambridge)

W. L. Brown. *Related Mathematics.* Pp. x, 290. 13s. 6d. 1932. (John Wiley; Chapman and Hall)

A. Buckley and C. F. G. Macdermott. *Elementary Mechanics.* Pp. viii, 216, xx. 4s. 1932. (Bell)

G. Cantor. *Gesammelte Abhandlungen.* Pp. vii, 486. RM. 48. 1932. (Springer, Berlin)

C. Carathéodory. *Conformal Representation.* Pp. viii, 106. 6s. 6d. 1932. Cambridge Tracts, 28. (Cambridge)

C. V. Durell. *Advanced Algebra.* I. Pp. viii, 194, xxii. 4s. 1932. (Bell)

J. W. M. Gunn. *A Geometry for Advanced Division, Central and Secondary Schools.* Part II. With Answers. Pp. x, 125-240. 2s. 1932. (Rivingtons)

G. A. Hanby. *Geometrical Drawing.* Pp. vii, 239. 6s. 1932. (Pitman)

H. Hancock. *Foundations of the Theory of Algebraic Numbers.* I. Pp. xxvii, 602. $8.00. 1931. (The Macmillan Company, New York)

D. Hilbert. *Gesammelte Abhandlungen. I. Zahlentheorie.* Pp. xiv, 539. RM. 48 1932. (Springer, Berlin)

W. Johnson. *Geometry for Senior Schools.* I. Pp. 72. II. Pp. 68. III. Pp. 64. 1s. each. 1932. (Oxford University Press)

E. Kamke. *Einführung in die Wahrscheinlichkeitstheorie.* Pp. vii, 182. Geh. RM. 10. Geb. RM. 11.50. 1932. (Hirzel)

P. S. de Laplace. *Philosophischer Versuch über die Wahrscheinlichkeit.* Ostwald's Klassiker der Exakten Wissenschaften, Nr. 233. Pp. vii, 211. RM. 9.60. 1932. (Akademische Verlagsgesellschaft, Leipzig)

D. Larrett and J. J. Walton. *Elementary Mechanics and Hydrostatics.* With Answers. Pp. 268. 3s. 6d. 1932. (Harrap)

R. K. Lochhead. *Valuation and Surplus.* Pp. xiii, 99. 7s. 6d. 1932. (Cambridge)

D. N. Mallik. *The Elements of Astronomy.* Second edition. Pp. 234. 14s. 1931. (Cambridge)

H. McKay. *Simplified Algebra.* Pupil's books: I. Pp. 64; II. Pp. 64; III. Pp. 64. 1s. 6d. each. Teacher's books: I. Pp. 108; II. Pp. 100; III. Pp. 96. 2s. 6d. each. 1932. (Oxford University Press)

R. L. Moore. *Foundations of Point Set Theory.* Pp. vii, 486. $5.00. 1932. American Math. Soc. Colloquium Publications, 13. (American Mathematical Society)

E. N. Mozley. *Graduated Exercises in Elementary Mathematics.* Teacher's edition. Pp. xv, 78. 2s. Pupil's edition. 1s. 3d. 1932. (Bell)

J. von Neumann. *Mathematische Grundlagen der Quantenmechanik.* Pp. 262. Geh. RM. 18. Geb. RM. 19.60. 1932. Grundlehren der math. Wiss., 38. (Springer)

E. P. Oakes. *A School Co-ordinate Geometry.* Pp. xiv, 265. 4s. 6d. 1932. (Pitman)

H. P. Pettit and **P. Luteyn.** *College Algebra.* Pp. vi, 284. 12s. 1932. (John Wiley and Sons ; Chapman and Hall, London)

J. F. Ritt. *Differential Equations from the Algebraic Standpoint.* Pp. x, 172. $2.50. 1932. American Math. Soc. Colloquium Publications, 14. (American Mathematical Society)

F. A. J. Rivett. *A New Junior Arithmetic.* Pp. 144. Without answers, 2s. With answers, 2s. 6d. 1932. (Edward Arnold)

D. E. Rutherford. *Modular Invariants.* Pp. viii, 84. 6s. 1932. Cambridge Tracts, 27. (Cambridge)

Sir Herbert Samuel. *Philosophy and the Ordinary Man.* Pp. 38. 1s. 6d. 1932. Presidential Address to the British Institute of Philosophy, 1932. (Kegan Paul, Trench, Trubner)

H. W. Turnbull and **A. C. Aitken.** *An Introduction to the Theory of Canonical Matrices.* Pp. xiii, 192. 17s. 6d. 1932. (Blackie)

J. H. Van Vleck. *The Theory of Electric and Magnetic Susceptibilities.* Pp. xii, 384. 30s. 1932. (Oxford ; at the Clarendon Press)

O. Veblen and **J. H. C. Whitehead.** *The Foundations of Differential Geometry.* Pp. ix, 96. 6s. 6d. 1932. Cambridge Tracts, 29. (Cambridge)

Notes on the Preparation of Mathematical Papers. Pp. 20. 1s. 1932. (Published for the London Mathematical Society by C. F. Hodgson and Son)

The Fifth Yearbook of the National Council of Teachers of Mathematics. The Teaching of Geometry. Pp. x, 206. $1.75. 1930. (Columbia University, New York)

The Sixth Yearbook of the National Council of Teachers of Mathematics. Mathematics in Modern Life. Pp. ix, 195. $1.75. 1931. (Columbia University, New York)

The Seventh Yearbook of the National Council of Teachers of Mathematics. The Teaching of Algebra. Pp. ix, 180. $1.75. 1932. (Columbia University, New York)

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This is under the direction of Mr. A. S. Gosset Tanner, M.A., Derby School, Derby, to whom all inquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the text-books on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. The names of those sending the questions will not be published.

The Mathematical Association

PROGRAMME

OF THE

ANNUAL MEETING

January 5th and 6th, 1933

THURSDAY AFTERNOON, 5TH JANUARY, 1933

2.15 p.m. Business.

3.45 p.m. The Presidential Address :—The Marquis and the Land Agent; a tale of the eighteenth century. Prof. G. N. Watson, Sc.D., F.R.S.

4.45 p.m. Interval.

5.15 p.m. The Precession of the Equinoxes. Dr. W. M. Smart.

FRIDAY, 6TH JANUARY, 1933

10.0 a.m. Side Tracks in Elementary Mathematics. Mr. F. C. Boon.

10.45 a.m. Parametric Equations in Elementary Analytical Geometry. Mr. A. Robson.

11.15 a.m. Methods of learning Geometrical Theorems. Mr. A. W. Siddons.

2.15 p.m. Discussion on The Study of Statistics in a School Course. To be opened by Mr. F. Sandon, followed by Mr. C. O. Tuckey, Dr. J. Wishart and Mr. R. M. Wright.

3.15 p.m. Relative Velocity. Mr. H. E. Piggott.

3.45 p.m. Lewis Carroll — Mathematician. Rev. D. B. Eperson.

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